

Marginal versus average beta of equity under corporate taxation

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Abstract

Even for fully equity-financed firms there can be substantial effects of taxation on after-tax costs of capital when there are depreciation deductions. Among the few studies of these effects, even fewer identify all effects correctly. Some claim to characterize the cost of capital, but fail to identify the marginal investment. When this is taxed together with inframarginal, marginal beta differs from average. This study shows the relation between the two and derives a correctly tax-adjusted weighted average cost of capital. To find asset betas, observed equity betas should not only be unlevered, but also “unaveraged” and “untaxed.”

KEYWORDS: Cost of capital, WACC, loss offset, tax shields, options

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1 Introduction

This paper explores the riskiness of depreciation tax shields and the consequences for the cost of capital after and before taxes. It finds substantial tax effects on the after-tax cost of equity and through this on the weighted average cost of capital (WACC). These effects, occurring even under full equity financing, have mostly been neglected in the literature. Among the few papers that study them, most do not identify the marginal investment, and thus cannot correctly identify the cost of capital.

The present paper uses valuation by components, known as Adjusted Present Value since Myers (1974). With this method, the WACC is unnecessary. Depreciation tax shields could be valued separately. There are at least two reasons to consider the WACC, nevertheless. One is that a WACC based on the Capital Asset Pricing Model (CAPM) is the most widespread capital budgeting technique (Graham and Harvey, 2001), and is likely to be so for some time to come. It is interesting to know how the equity component in the WACC should be tax adjusted, and what mistakes are being made. The other reason is that observed betas from equity markets are sometimes being used as basis for finding asset betas (see, e.g., Brealey, Myers, and Allen (2008), p. 482). In addition to the need to “unlever” the observed betas, the present paper points out the need to “untax” them when there are depreciation tax shields, and to “unaverage” them when, also, the marginal investment is being taxed together with inframarginal.

In the first part of the present paper it is assumed that depreciation tax shields are risk free, i.e., the firm is always in tax position and will thus earn the tax value of depreciation deductions with certainty and without delays. Under this assumption the beta of equity is approximately proportional to $(1-t)$, i.e., one minus the corporate tax rate. This is exactly true (see Proposition 1 below) under those tax systems which allow interest accumulation

on investment-related deductions, so that the present discounted value of deductions is equal to the investment.

The intuition behind the result is best seen with a pure cash flow tax (Brown 1948) as a point of reference. This is a proportional tax on non-financial cash flows, with payout of negative taxes in years with negative net cash flows.¹ The beta of the after-corporate-tax cash flow is unaffected by a pure cash flow tax. The tax acts cash-flow-wise as just another shareholder. Consider next what happens if negative taxes are not paid out, but postponed and given as tax deductions with interest in later years.² This change from the Brown tax is like a risk free loan from the firm to the tax authorities (lending by the firm), and acts risk-wise as the opposite of traditional leverage (borrowing by the firm). The systematic risk of the after-tax cash flow is reduced, i.e., multiplied by the factor $(1 - t)$.

A change from immediate expensing (in the Brown tax) to depreciation deductions has almost the same risk reduction effect as the change which was just described. But typical depreciation deductions have a somewhat lower discounted present value, and thus reduce risk somewhat less. The precise expressions are derived in this paper.

The cost of equity and, more generally, the cost of capital are defined as required expected rates of return. This refers to a marginal investment. There are two reasons why the present paper considers a production function with decreasing returns to scale (DRS), and then determines the marginal investment endogenously, depending on taxes, pre-tax riskiness, and other parameters of the problem. The first reason is that the existing literature has been unclear on the need to examine a marginal investment. The model below shows that a tax system with depreciation tax shields is all one needs to get a difference between the marginal and average beta of equity, and thus a difference between the required, marginal expected rate of return and the average expected rate of return (both after-tax). This invalidates studies which claim to find the cost of capital under taxation

without identifying the marginal investment. The other reason is that DRS allows an analytical approach to quantifying the riskiness of the tax shields.

Although some authors have regarded depreciation tax shields as essentially risk free, a more realistic model is interesting when the purpose is to explore the risk-reducing effect of these tax shields. This is achieved in section 6 by using the analogy between the tax claim and a call option.³ Admittedly, the model of risky tax shields is based on a number of simplifying assumptions, and does not include debt financing. It nevertheless gives useful insights into the riskiness of the tax and after-tax cash flows, such as which variables are important and which have no impact.

The paper is organized as follows: Section 2 reviews the previous literature. Section 3 presents the model of valuation and DRS production. Section 4 introduces the tax system and gives results for the all-equity case with risk free tax shields. Section 5 combines risk free debt and risk free tax shields and finds the WACC with tax adjusted cost of equity. Section 6 considers risky tax shields and compares results with those from section 4, both in tables and diagrams. Section 7 has results on the required expected return before taxes. Section 8 extends the model to allow for entry costs, possibly outweighing the rent in DRS production. Section 9 contains additional discussion of some aspects of the model. Section 10 concludes. Some proofs and additional details are in the appendices.

2 The existing literature

The previous literature on the topic is scattered in public and financial economics, and there are some predecessors which focus on natural resource extraction. In public economics there is a substantial literature on the effect of taxes on the cost of capital before taxes. In King and Fullerton (1984, p. 10) this is formulated as $p = c(r)$, where p is the real cost

of capital, r is the real market interest rate, and the function c “depends upon details of the tax code.” In most of this literature there is no consideration of uncertainty, and r is taken as given in a partial equilibrium model. Under uncertainty this is inadequate (and easily misleading), even in a similar partial equilibrium framework. There is no single rate of return which can play the role of r . Public economics has the advantage, however, that the studies typically realize the need to identify the marginal project. The relationship to Hall and Jorgenson (1969) is shown in section 7 below.

A well-known textbook in finance, Brealey, Myers, and Allen (2008), ignores the possibility to say something systematically about how the cost of equity depends on depreciation and similar tax shields. They state⁴ (p. 561) that “Depreciation tax shields contribute to project cash flow, but they are not valued separately; they are just folded into project cash flows along with dozens, or hundreds, of other specific inflows and outflows. The project’s opportunity cost of capital reflects the average risk of the resulting aggregate.” This practice obscures the tax-induced difference between marginal and average betas. Moreover, it is unfortunate if the firm operates under different tax systems.

DeAngelo and Masulis (1980) analyze the effect of taxes on the financing choice of a firm, and bring in the effects of non-debt tax shields. Their main concern is with the substitution of such tax shields for debt tax shields. They do not analyze the direct effect of non-debt tax shields on the beta of equity.⁵

Eight previous theoretical studies which discuss the effect of taxes on the risk of after-tax rates of return, allowing for tax effects even in the absence of debt, are Levy and Arditti (1973), Galai (1988), Jacoby and Laughton (1992), Derrig (1994), Bradley (1998), Galai (1998), Lund (2002a), and Rao and Stevens (2006). Both Levy and Arditti (1973) and Lund (2002a) determine the marginal investment to find the required expected rate of return, while the others do not. More details are given in section 4 below.

Four of the eight studies assume that firms always pay taxes, so that tax shields are risk free. The last two on the list do not. Earlier, Jacoby and Laughton (1992) and Bradley (1998)⁶ use the finance-theoretic approach originally developed for option valuation to study valuation of natural resource projects under taxation. They use Monte-Carlo simulations for specific resource extraction projects, extending the APV method. To find project values after tax there is no need for required expected returns for after-tax net cash flows. But these are found after the net values of cash flows have been calculated, technically like internal rates of return. The present paper highlights, through an analytical model, some typical mechanisms behind these results.

Galai (1998) is, together with Lund (2002a), the paper most closely related to the present one. Galai has a theoretical two-period model with results on the systematic risk of the cash flows to the three claimants, equity, debt, and tax authorities, and on possible conflicts of interest between these. The equity beta is found to be declining in the tax rate, but the required expected return is not determined. Lund (2002a) has a model very similar to the present paper, but only considers the marginal investment.

More recently, like the present paper, Rao and Stevens (2006) set up a two-period model of a firm subject to taxation, with investment in the first and a risky outcome in the second period. Like in the present paper, the priced risk is determined by the covariance with an exogenously given process which is unaffected by the tax system to be analyzed. Their model is more general by considering risky debt, and in some other respects. The pricing model is an approximate Arbitrage Pricing Theory (APT), which is robust with respect to different distributional assumptions. The firm in their model can be solvent or insolvent, in tax position or not, and if in tax position, using two different tax shields, debt and non-debt, partly or fully. They improve upon the literature by a simultaneous solution to the cost of debt, the optimal level of debt, and the risks of the tax shields. From this

follow also the WACC and the risks and values of the different cash flows, including the tax claim. But they have a different focus for their analysis from that of the present paper.⁷ They give no results for effects on their endogenous variables of changes in tax rates or other tax parameters. Their model starts with some exogenous project profitability before tax, and does not determine the risk of the tax shields endogenously based on the taxation of the marginal investment together with inframarginal investment. The model becomes quite complicated and cannot be solved analytically.⁸

Summers (1987) investigates the riskiness of depreciation deductions, and finds that they have low systematic risk. He recognizes that firms in many cases discount these tax shields too heavily, and states that “patterns of investment may be very substantially distorted in ways not considered in standard analyses of the effects of tax incentives” (p. 302). He goes on to consider consequences for tax reform. Gordon and Wilson (1989) (fn. 10) mention that depreciation deductions are “normally riskfree in nominal terms.” One message of the present paper is that before one concludes that these tax shields can be regarded as risk free, one should carefully consider under what circumstances they will be somewhat risky.

This paper can also be seen as a supplement to the empirical work on estimating marginal tax rates of firms taking tax carry-forward and carry-back into consideration. Some central references are Auerbach and Poterba (1987), Shevlin (1990), Graham (1996), and Shanker (2000). While the empirical studies are more realistic by taking multi-period effects into account, the present model gives analytical solutions, identifying which factors are likely to have important effects.

3 The model

A firm invests in period 0 and produces in period 1, only. The firm considers an investment project with decreasing returns to scale. It is free to choose the scale of investment, and uses an APV-based method. The optimal choice is endogenous, determined by the tax system and other parameters in each case below. In this way the minimum required expected return to equity in each case is determined. The first assumption of the model is:

Assumption 1: *The firm is fully equity financed and maximizes its market value according to the Capital Asset Pricing Model,*

$$E(r_i) = r + \beta_i[E(r_m) - r], \quad (1)$$

where $r > 0$.

All variables are nominal. The model is consistent with deterministic inflation, whereas stochastic inflation would require a more complicated model, especially if taxes are not inflation adjusted.

When changes in tax rates or deductions are considered below, these are assumed not to affect the capital market equilibrium. This will be a good approximation if they apply in small sectors of the economy (e.g., natural resource extraction), or abroad in economies (“host countries”) which are small in relation to the domestic one (the “home country”). This is thus a partial equilibrium analysis.⁹

The (“home”) economy where the firm’s shares are traded may have a tax system, which is exogenously given and fixed throughout the analysis, and possibly¹⁰ reflected in r . A consequence of the CAPM is that the claim to any uncertain cash flow X , to be received in period 1, has a period-0 value of

$$\varphi(X) = \frac{1}{1+r}[E(X) - \lambda \text{cov}(X, r_m)], \quad (2)$$

where $\lambda = [E(r_m) - r]/\text{var}(r_m)$. Equation (2) defines a valuation function φ to be applied below.

Results which follow rely on a constant “asset beta.” With the model’s simple cash flows before tax, this is simply the beta of a claim to one unit of output, defined in relation to the return $P/\varphi(P)$,

$$\beta_P = \frac{\text{cov}\left(\frac{P}{\varphi(P)}, r_m\right)}{\text{var}(r_m)}. \quad (3)$$

Assumption 2: *In period 0 the firm invests an amount $I > 0$ in a project. In period 1 the project produces a quantity Q to be sold at an uncertain price P . The joint probability distribution of (P, r_m) is exogenous to the firm, and $\text{cov}(P, r_m) > 0$. There is no production flexibility; Q is fixed after the project has been initiated. There is no salvage value and no operating cost in period 1.*

The assumption of $\text{cov}(P, r_m) > 0$ can easily be relaxed. It is only a convenience to simplify the verbal discussions below.

4 Case F: Tax deductions are risk free

This section will arrive at two expressions for the beta of equity under the assumption that the firm is certain to be in tax position in the next period, Case F (F for risk Free). The two betas will be referred to as the marginal and average beta. Section 6 will arrive at two other expressions for the beta of equity, under the assumption that the firm is uncertain whether it will pay taxes next period. These two betas will also be referred to as the marginal and average beta, for that case. In addition to these four betas there will be a reference in section 6 to the beta of equity for a marginal project taxed alone, under

uncertainty about the tax position, as derived in Lund (2002a). There is also a generalized version of the model in section 8.

Assumption 3: *A tax at rate $t \in [0, 1)$ will be paid with certainty in the production period. The tax base is operating revenue less cI . There is also a tax relief of taI in period 0. The constants a and $c/(1+r)$ are in the interval $[0, 1]$; moreover, $t[a + c/(1+r)] < 1$.*

This general formulation allows for accelerated depreciation with, e.g., $a > 0$ and $a + c = 1$, or a standard depreciation interpreted (since there is only one period with production) as $a = 0, c = 1$. The requirement $t[a + c/(1+r)] < 1$ precludes “gold plating incentives,” i.e., the tax system carrying more, in present value terms, than one hundred percent of an investment cost.¹¹

Assumption 3 implies that a negative tax base gives a negative tax paid out by the authorities. While this is unrealistic for most tax systems when the project stands alone, it is often a good approximation when the marginal project is added to other activity which is more profitable and only weakly correlated with it. An alternative assumption for the second period is considered in section 6. For the first period, however, no alternative is considered.

In the Case FM (M for marginal) of a marginal project alone, the cash flow to equity in period 1 is

$$X_{FM} = PQ(1 - t) + tcI. \quad (4)$$

For each set of tax and other parameters, Q/I is set so that the project is exactly marginal after tax. This does not lead to an optimal scale of investment. The purpose is to characterize marginal investment. Technically this is a project with constant returns to scale (CRS). The market value in period 0 of a claim to this is

$$\varphi(X_{FM}) = \varphi(P)Q(1 - t) + \frac{tcI}{1 + r}. \quad (5)$$

For a marginal project the expression must be equal to the financing need after taxes, $I(1 - ta)$, so that Q/I is determined by

$$I(1 - ta) = \varphi(X_{FM}) = \varphi(P)Q(1 - t) + \frac{tcI}{1 + r}, \quad (6)$$

which implies

$$\varphi(P)Q(1 - t) = I \left(1 - ta - t \frac{c}{1 + r} \right). \quad (7)$$

The beta of equity is a value-weighted average of the betas of the elements of the cash flow. From (4) this is,

$$\beta_{FM} = \frac{\varphi(P)Q(1 - t)}{\varphi(P)Q(1 - t) + It \frac{c}{1 + r}} \beta_P = \frac{1 - ta - t \frac{c}{1 + r}}{1 - ta} \beta_P, \quad (8)$$

where the second equality follows from (7) above.¹² This can be summarized as follows:

Proposition 1: *Under Assumptions 1–3, the beta of equity for a marginal investment with constant returns to scale is given by (8). When $tc > 0$, it is strictly decreasing in the tax rate t , in the investment tax credit rate a , and in the present value of the deduction rate c .*

The proof is in Appendix A. The beta of equity is decreasing in the tax rate under any tax system with postponed deductions for investment outlays. Under a pure cash flow tax ($a = 1, c = 0$) there is no such effect of the tax rate. If the investment-related tax shields appear in period 1, but have a period-0 present value equal to the investment ($a = 0, c = 1 + r$), equation (7) implies that the marginal investment will be unaffected by the tax rate. But the beta of equity will be $(1 - t)\beta_P$. This is the case mentioned in the introduction with a Brown tax as a point of departure, then a postponement with interest accumulation. The suggestion in the introduction that the beta of equity is approximately equal to $(1 - t)\beta_P$ holds when a is small relative to $\frac{c}{1 + r}$. In particular, $a = 0$ gives $\beta_{FM} = (1 - t \frac{c}{1 + r})\beta_P$, which is close to $(1 - t)\beta_P$ when $\frac{c}{1 + r}$ is close to unity.

Consider now the DRS Case, FA (A for average). Instead of technically adjusting Q to find the characteristics of a marginal project, there is now a first-order condition which determines I .

Assumption 4: *Produced quantity is $Q = f(I) = \omega I^\alpha$. The production function f has $\omega > 0$, $\alpha \in (0, 1)$.*

The cash flow to equity in period 1 is

$$X_{FA} = Pf(I)(1 - t) + tcI. \quad (9)$$

The market value of a claim to this is

$$\varphi(X_{FA}) = \varphi(P)f(I)(1 - t) + \frac{tcI}{1 + r}. \quad (10)$$

The firm chooses the optimal scale to maximize

$$\pi_F(I) = \varphi(X_{FA}) - I(1 - ta). \quad (11)$$

The first-order condition for a maximum is

$$\varphi(P)f'(I) = \frac{1 - ta - t\frac{c}{1+r}}{1 - t}, \quad (12)$$

which can be rewritten, based on the analytical production function, as

$$\varphi(P)f(I)(1 - t) = \frac{I}{\alpha} \left(1 - ta - t\frac{c}{1+r} \right). \quad (13)$$

The beta of equity is a value-weighted average of the betas of the elements of the cash flow. From (9) this is

$$\beta_{FA} = \frac{\varphi(P)f(I)(1 - t)}{\varphi(P)f(I)(1 - t) + It\frac{c}{1+r}} \beta_P. \quad (14)$$

According to (13), the optimal ratio $I/f(I)$ is proportional to α (since the other variables appearing in (13) are exogenous). Consider as a thought experiment what happens

when the exogenous α is reduced from unity (which is its implicit value in (7)). The relative weight of the last term in (10), and in the denominator in (14), is reduced, and β_{FA} will get closer to the before-tax β_P .

The first-order condition and the parameterized production function together give

$$\beta_{FA} = \frac{1 - ta - t\frac{c}{1+r}}{1 - ta - t\frac{c}{1+r}(1 - \alpha)}\beta_P, \quad (15)$$

which again is decreasing in the tax rate as long as $c > 0$. As α approaches unity (i.e., CRS), β_{FA} approaches β_{FM} . The result can be summarized as follows:

Proposition 2: *Under Assumptions 1–4, the beta of equity is given by (15). When $tc > 0$, it is strictly decreasing in the tax rate t , in the investment tax credit rate a , in the present value of the deduction rate c , and in the scale elasticity α .*

The proof is in Appendix A. Observe that $\beta_{FM} < \beta_{FA}$ when $tc(1 - \alpha) > 0$. The reader may verify that in this case, the ratio β_{FM}/β_{FA} is decreasing in the tax rate (t), the investment tax credit rate (a), the deduction rate (c), and in $(1 - \alpha)$.

The two different expressions for the beta of equity will be called *marginal beta* and *average beta*, respectively. They are both relevant as descriptions of systematic risk for the same project. The average beta will describe the systematic risk of the project as a whole, and in particular, the systematic risk of the shares in a firm with only this project. The marginal beta is still the relevant one for decision making at the margin, which may be decentralized within the firm. The correct beta for calculating the required expected rate of return is the marginal beta. The reason is that at the margin, the ratio Q/I is given by (7), not by (13). The result on the appropriate beta for decision-making is:

Proposition 3: *Under Assumptions 1–4, the firm's optimal investment can be found by maximizing its expected present value with a constant risk-adjusted discount rate based on the beta from (8). The same optimum can be found from maximizing the expected present*

value with a non-constant risk-adjusted discount rate based on the beta from (14), with beta being a function of the investment level, I .

The proof is in Appendix A. If the average beta, β_{FA} , is used, it can not be considered a constant. Its value will be endogenously determined as part of the optimization. This restricts its usefulness from a managerial point of view.

Proposition 3 has important implications for all studies which consider the effect of taxation on after-tax returns to equity. Of the eight immediate predecessors of this study, mentioned in section 2, only Levy and Arditti (1973) and Lund (2002a) identify tax effects on the *required* expected rate of return, i.e., the cost of capital, after tax.¹³ Some authors (Galai 1998, Rao and Stevens 2006) have studied tax effects for any exogenously given level of profitability,¹⁴ and some (Jacoby and Laughton 1992, Bradley 1998) have studied the same for specific numerical examples, with various realistic (or “reasonable”) profitability levels.¹⁵ This works well when the aim is to find tax effects on the systematic risk of a given project. But if one wants the effect on the required expected rate of return, one needs to consider a project which is exactly marginal after tax.

The reason why the marginal project and the DRS project have two different betas is not that one of them is located outside the security market line (SML), which is a problem sometimes seen in similar analyses. Both are on the SML because of the way their betas are determined in (8) and (14), as value-weighted averages of betas of correctly valued assets. The reason is instead that the period 1 cash flows of the two projects are composed differently. The only reason for this is the tax system, since there are no operating costs or other elements in those cash flows apart from the output values and the tax shields. The phenomenon occurs because the tax system allows investment-related deductions, based on investment which is not equal to the valuation of the subsequent project cash flow. Equation (6), $I(1 - ta) = \varphi(X_{FM})$, does not hold for the DRS project.

The term “cost of capital” is generally used for a minimum required expected rate of return. This indicates that the term should not be used for a rate of return of a DRS project which yields supranormal profits (rents). However, the expected return from equations (9) and (10), $E(X_{FA})/\varphi(X_{FA})$, is an equilibrium expected return, and does not in itself exhibit any supranormal profit. This expected return will reflect the systematic risk quantified by β_{FA} . It is (one plus) the correct risk-adjusted discount rate to be used for finding the market value of X_{FA} , but only for some given ratio $tcI/E(P)f(I)(1-t)$. In the model this is optimally chosen and depends on α .

Proposition 3 thus demonstrates the need to “unaverage” observed betas. This is not much different from a correction for unusual operating leverage, except that the correction depends totally on the tax system and goes in the opposite direction: Higher investment cost implies lower beta, while higher operating cost typically would imply higher beta of the net future cash flow. A possibly realistic extension of the model would be to assume that an optimal DRS project would have an (average) operating leverage which is different from that of a marginal project. But this possible complication, which could exist independently of taxes, does not eliminate the tax effect highlighted by the present model.

5 Case B: Borrowing; weighted average cost of capital

The weighted average cost of capital (WACC) will be derived for the case of risk free debt in a fixed ratio to equity, and risk free tax shields. First the model of the previous sections is extended to allow for risk free debt. Both the marginal and average beta are derived. The marginal beta of equity is the basis for defining the WACC in the standard way. Then it is shown that this WACC is the correct discount rate for the investment decision. Assumption 1 is replaced by:

Assumption 5: *The firm maximizes its market value according to a possibly tax-adjusted Capital Asset Pricing Model,*

$$E(r_i) = r + \beta_i[E(r_m) - r], \quad (16)$$

where $r > 0$ may differ from the risk free interest rate by a tax adjustment factor. A fraction $\eta \in (0, 1]$ of the firm's after-tax financing need in period 0 is financed by equity and the remainder by debt, B . Debt is repaid with interest with full certainty in period 1. The borrowing rate is r_b . A fraction $g \in [0, 1]$ of the firm's interest payment is tax deductible.

Since r in the previous sections only plays the role of the intercept in the CAPM equation, this definition is maintained, whereas a new variable is defined for the borrowing rate. There will be a tax advantage of debt if $r_b(1 - tg) < r$. In a Miller (1977) equilibrium, this is instead satisfied with equality, because of the tax disadvantage of debt on the hand of the marginal investor. The possible tax adjustment in r originates in the home country or the world capital market, so r is not affected when changes in t are considered.

With a fixed debt/equity ratio $(1 - \eta)/\eta$ and a DRS project, the cash flow to equity in period 1 is

$$X_{BA} = Pf(I)(1 - t) + tcI - (1 + r_b)B + tgr_bB, \quad (17)$$

where $B = (1 - \eta)I(1 - ta)$. The market value of a claim to this is

$$\varphi(X_{BA}) = \varphi(P)f(I)(1 - t) + \frac{tcI}{1 + r} - \frac{(1 + r_b(1 - tg))(1 - \eta)(1 - ta)I}{1 + r}. \quad (18)$$

The firm chooses the optimal scale to maximize

$$\pi_B(I) = \varphi(X_{BA}) - \eta I(1 - ta). \quad (19)$$

The first-order condition for a maximum is

$$\varphi(P)f'(I)(1 - t) = (1 - ta) \left[\eta + (1 - \eta) \frac{1 + r_b(1 - tg)}{1 + r} \right] - \frac{tc}{1 + r}, \quad (20)$$

which replaces equation (12) above. Define the expression in square brackets as

$$\Lambda \equiv \eta + (1 - \eta) \frac{1 + r_b(1 - tg)}{1 + r}. \quad (21)$$

This measures the relative tax advantage of debt at this debt/equity ratio. If $\Lambda = 1$, there is no such advantage, due to $\eta = 1$ or $r_b(1 - tg) = r$ or both. If $\eta < 1$ and $r_b(1 - tg) < r$, there is the well-known tax advantage of debt. When $\eta < 1$, Λ is decreasing in the tax rate t .

The following equation and result are shown in Appendix B.

$$\beta_{BA} = \frac{(1 - ta)\Lambda - \frac{tc}{1+r}}{(1 - ta)\Lambda - \frac{tc(1-\alpha)}{1+r} - \alpha(\Lambda - \eta)(1 - ta)} \beta_P. \quad (22)$$

Proposition 4: *Under Assumptions 2–5, with Λ defined as in (21), the beta of equity is given by (22). When $c > 0$ and/or $\eta < 1$, it is strictly decreasing in the tax rate t .*

The definitions which follow in (23) and (24) are justified by the next Proposition 5. Define the marginal beta in case B by taking the CRS limit,

$$\beta_{BM} = \lim_{\alpha \rightarrow 1} \beta_{BA} = \frac{1}{\eta} \left[\Lambda - \frac{tc}{(1 + r)(1 - ta)} \right] \beta_P. \quad (23)$$

The equity ratio η appears in Λ , but its strongest effect on β_{BM} comes through its appearance in the denominator of the first fraction. As is well known, a low equity ratio will increase the systematic risk of equity,¹⁶ and it will thus counteract the risk-reducing effect of the investment-related deductions, which are the focus of the present paper, expressed in the tc term. However, for some given η and typical depreciation deductions, the effect of increasing the tax rate will still be a strong reduction in beta.

Use β_{BM} to define the WACC in the standard way,¹⁷

$$r_w = \eta \{r + \beta_{BM} [E(r_m) - r]\} + (1 - \eta)r_b(1 - tg). \quad (24)$$

Applying this as discount rate for the investment decision means to maximize

$$\frac{E(P)f(I)(1-t) + tcI}{1+r_w} - I(1-ta). \quad (25)$$

The following proposition, which is shown in Appendix B, shows that this maximization gives the correct answer.

Proposition 5: *Under Assumptions 2–5, the maximization of (25), with the WACC defined by (23) and (24), gives the same solution as in equation (20).*

This shows the relevance of the marginal beta for investment decisions, also when there is borrowing. According to the survey by Graham and Harvey (2001), more than 70 percent of U.S. firms apply the CAPM to find the beta of equity, which is then entered into a WACC. This section has shown how the relevant beta of equity depends on the corporate tax system levied on the firm’s activity, when that system allows depreciation deductions. If the beta of equity is based on observations from the stock market, it should not only be unlevered to find an asset beta. It should also be “unaveraged” to find the marginal beta, since the observed beta typically is an average beta. It should furthermore be “untaxed” to correct for the tax system, in particular if the WACC is being applied under different systems or rates of taxes and/or deductions.

6 Case R: Uncertain tax position

The results for Case F above are based on the assumption that the firm is certain to be in tax position in period 1. While the tax element tPQ is perfectly correlated with the operating revenue, the depreciation deductions were assumed to be risk free.

To get analytical results the present section assumes that there is no debt and no loss offset at all. This means that the two-period model is taken literally and the tax code does

not allow carry-backs. One purpose of the present paper is to see how much the results of Case F are modified when tax shields are risky. Thus it is relevant to consider this most extreme riskiness.¹⁸ It turns out that even then, the beta of equity is substantially lower than the asset beta before taxes, given reasonable parameter values. The cash flow to equity in period 1 is

$$PQ - t \max(0, PQ - cI). \quad (26)$$

Lund (2002a) arrived at an analytical solution for marginal beta in this case under the assumption that the marginal investment constitutes the whole tax base for the firm.¹⁹

A marginal beta may now take different meanings. A more realistic marginal beta recognizes that the marginal project is typically part of a larger activity, and that the probability of being in tax position depends on the outcome of that larger activity. This will be analyzed in line with the model of the previous sections: The larger activity consists of a DRS investment project, the output of which is being sold at a single stochastic price in the single future period.

Let Case R (for Risky deductions) denote the case with an uncertain tax position. The following assumption replaces Assumption 3 above:

Assumption 6: *The tax base in period 1 is operating revenue less cI . When this is positive, there is a tax paid at a rate t . When it is negative, the tax system gives no loss offset at all. There is also a tax relief of taI in period 0. The constants a and $c/(1+r)$ are in the interval $[0, 1]$; moreover, $t[a + c/(1+r)] < 1$.*

The tax cash flow is similar to a cash flow from a European call option. McDonald and Siegel (1984) show how to value this option when the underlying asset has a rate-of-return shortfall. The valuation of the non-linear cash flow is specified as follows:

Assumption 7: A claim to a period-1 cash flow $\max(0, P - K)$, where K is any positive constant, has a period-0 market value according to the model in McDonald and Siegel (1984). The value can be written as

$$\varphi(P)N(z_1) - \frac{K}{1+r}N(z_2), \quad (27)$$

where

$$z_1 = \frac{\ln(\varphi(P)) - \ln(K/(1+r))}{\sigma} + \sigma/2, \quad z_2 = z_1 - \sigma, \quad (28)$$

N is the standard normal distribution function, and σ is the instantaneous standard deviation of the price.²⁰ To apply an absence-of-arbitrage argument for option valuation when there is a rate-of-return shortfall, forward or futures contracts for the output must be traded, or there must exist traded assets which allow the replication of such contracts. The validity of an option valuation formula in an economy with taxation is discussed, e.g., in McDonald (2006), p. 341. He concludes that “When dealers are the effective price-setters in a market, taxes should not affect prices.”

The combination of the CAPM and the option pricing model relies on, e.g., the assumptions in Galai and Masulis (1976).²¹ The CAPM will now be a single-beta version of the intertemporal CAPM of Merton (1973). Capital markets operate in continuous time, whereas investment, production and taxes happen at discrete points in time.

In what follows it is assumed that the exogenous variables β_P and σ can be seen as unrelated as long as $\sigma > 0$, cf. footnote 14 in McDonald and Siegel (1986). A change in σ could be interpreted as, e.g., additive or multiplicative noise in P , stochastically independent of the previous (P, r_m) .²²

Propositions 6–8 are shown in Appendix C:

Proposition 6: *Under Assumptions 1, 2, 4, 6, 7, the beta of equity is given by (29).*

$$\beta_{RA} = \frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{1 - ta - tN(z_{2D})\frac{c}{1+r}(1 - \alpha)}\beta_P, \quad (29)$$

where z_{2D} is given by

$$z_{2D} = \frac{1}{\sigma} \ln \left(\frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{\alpha[1 - tN(z_{2D} + \sigma)]\frac{c}{1+r}} \right) - \frac{\sigma}{2}. \quad (30)$$

Although this equation cannot be solved explicitly, it determines z_{2D} implicitly as function of t , a , $c/(1+r)$, σ , and α . The rate-of-return shortfall (or convenience yield) does not affect the ratio β_{RA}/β_P .

Proposition 7: *Under Assumptions 1, 2, 4, 6, 7, the beta for a marginal investment taxed together with the optimally chosen DRS investment is given by (31).*

$$\beta_{RM} = \frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{1 - ta}\beta_P. \quad (31)$$

This means that the relationship between marginal and average beta is similar to that of the previous case, which had full certainty about the tax position. There is an extra term containing $tc(1 - \alpha)$ subtracted in the denominator of the average beta.

The two equations (31) and (29) should be compared with (8) and (15). Clearly the effect of the uncertainty in the tax position is similar to a reduced tax rate in period 1, reflecting that the probability of earning the tax shields is less than one hundred percent.

For comparison, the marginal beta in the stand-alone CRS case can be found. This is denoted RC because it only applies if the project actually has constant returns to scale. The probability of being in tax position is lower in this case.

Proposition 8: *Under Assumptions 1, 2, 6, and 7, the beta for a marginal investment taxed alone is given by (32).*

$$\beta_{RC} = \frac{1 - ta - tN(z_{2C})\frac{c}{1+r}}{1 - ta}, \quad (32)$$

where z_{2C} is given by

$$z_{2C} = \frac{1}{\sigma} \ln \left(\frac{1 - ta - tN(z_{2C})\frac{c}{1+r}}{[1 - tN(z_{2C} + \sigma)]\frac{c}{1+r}} \right) - \frac{\sigma}{2}, \quad (33)$$

which is the limit of (30) as $\alpha \rightarrow 1$.

This is the case considered in Lund (2002a), except that equation (33) was not given there. Table I summarizes the five all-equity subcases. The rightmost column gives the ratio of β_i (the beta of equity) to β_P in each subcase i . TABLE

How the marginal and average betas depend on t, σ , and α has been traced through numerical solution to the non-linear equations. All cases considered have $a = 0$ and the ratio $c/(1+r)$ fixed at $1/1.05$. The central parameter configuration considered is $t = 0.35, \sigma = 0.3$. These are not unreasonable numbers (when the time unit is one year). For simplicity the verbal discussion below will assume $\beta_P = 1$. The five equity betas, divided by β_P , are shown in Figure I as functions of the scale elasticity α . A sixth relevant curve for comparison would be β_P itself, horizontal at 1.0 in the diagram. This would be the beta of equity without taxation or with pure cash flow taxation. FIGURE

Figure I shows that the betas have the expected properties. Consider first Case F with riskless tax shields. The sparsely dotted horizontal line gives the marginal β_{FM} , while the heavily/infrequently dashed curve gives the average β_{FA} . The first one is a constant, independent of α . The numerical value, approximately 0.67, is close to $(1-t)\beta_P$. The average beta declines from β_P to β_{FM} as α goes from zero to unity. The relationship is slightly convex. The upper limit, equal to β_P , comes from the fact that the relative weight on the final term in (10) goes to zero. In the limit as $\alpha \rightarrow 0^+$, the future cash flow is proportional to P and has the same systematic risk as P . The ratio β_{FM}/β_{FA} increases I HERE.

towards unity as $\alpha \rightarrow 1^-$, as mentioned above, because the whole project approaches a marginal project at this limit.

In Case F the effect on β_{FA} of varying α comes through the changing relative weights of two cash flow elements, one proportional to P , the other risk free. This effect is still present in Case R with risky tax shields. But here there is another, opposing effect: A higher α reduces the probability of being in tax position in period 1. This affects both marginal and average beta in Case R. The densely dashed curve gives the β_{RM} of the marginal investment taxed together with the inframarginal investment. This is increasing and convex as function of α . As $\alpha \rightarrow 1^-$, the technology approaches CRS, and the risk of the tax shields increases. The upper limit is thus equal to the β_{RC} of a marginal investment taxed alone. The lower limit, when $\alpha \rightarrow 0^+$, is equal to the marginal β_{FM} when the tax shields are risk free. In this limit there is so much income, relative to the investment, that the probability of not paying taxes goes to zero. The solid curve gives the average β_{RA} for the case of risky tax shields. For small α values there is no detectable difference between this and β_{FA} , since the risk is minuscule. As α increases towards unity, β_{RA} approaches β_{RM} from above, since the DRS investment approaches a CRS investment. The feature that β_{RA} is a nonmonotonic function of α is not so easy to explain (and may not be true for all parameter configurations).

The results on β_{RA} can be compared with those of Jacoby and Laughton (1992), although their numerical examples are more complicated, involving also various degrees of operating leverage. In their Figure 5 the systematic risk of the net after-tax cash flow decreases monotonically with increasing rent, which is consistent with the right-hand increasing part of the β_{RA} curve shown in Figure I here. Rents increase to the right in their Figure 5, to the left in Figure I here. The convexity is qualitatively the same in both curves. Their conclusion (p. 44) that “the larger fields will be undervalued relative to the

smaller fields if all are discounted with the same discounting structure, as they would be using standard DCF methods” is true within the range they cover, but not in general, due to the non-monotonicity demonstrated here.

Clearly, even the DRS case with risky tax shields can have betas substantially lower than β_P . In this case the marginal beta curve, β_{RM} , satisfies the intuition that it has less risk than the stand-alone marginal beta, β_{RC} , as an effect of being taxed together with an infra-marginal cash flow. But the average beta, β_{RA} , does not exhibit this property uniformly, and in fact, the difference between marginal and average beta is just as large in this case as in the case with risk free tax shields. The convexity of the curves strengthens the feature that tax shields, and thus after-tax equity, have relatively low systematic risk when there are moderately decreasing returns to scale (say, $0.6 < \alpha < 0.9$).

Figures II and III show some sensitivities to changes in the tax rate, t , and the volatility, σ . The three non-constant curves from Figure I are reproduced as (similarly) dotted curves, and the corresponding three curves for the new value of t or σ are drawn as dashed or solid. The values of the constant β_{FM} and β_{RC} are now only shown implicitly, as the endpoint values for the curves.

Figure II shows that all betas are decreased if the tax rate is raised (and vice versa), which was also the main point in Lund (2002a) for the cases considered there. The effect on the lowest values (β_{FM} , which is the limit of β_{RM} for low α , and of β_{FA} for high α) seems to be proportional to $(1 - t)$, which is almost correct when $c/(1 + r)$ is close to unity, see also Corollary 2.2 in Lund (2002a). For a given α , the ratio β_{FM}/β_{FA} is decreasing in t (i.e., the two betas differ more with higher t), as mentioned above. FIGURE II
HERE.

Figure III shows only one β_{FA} curve, as this is unaffected by a change in volatility. The figure shows that except for this, a lower σ works in the same direction as a higher t . But the effects of changes in σ are only discernible for higher values of α , and the magnitudes FIGURE III
HERE.

of the effects are not very large. The effects of σ on the ratio β_{FA}/β_P are robust results in the sense that they do not rely on any assumption about the relationship between σ and β_P . However, the effects of σ on β_{FA} (separately) could also include effects via possible changes in β_P , which have not been analyzed here.²³

7 Cost of capital before taxes

The cost of capital before corporate taxes is the traditional measure for the effects of the tax system on the acceptance or rejection of real (non-financial) investment projects. This determines the possible distortionary effects of the tax system, although the present paper does not discuss what would be the relevant basis for comparison in various circumstances.

The expected rate of return before corporate taxes, plus 1, is $E(P)Q/I$, which can be rewritten as

$$\frac{E(P)Q}{I} = \frac{E(P)}{\varphi(P)} \cdot \frac{\varphi(P)Q}{I}. \quad (34)$$

Of the two fractions on the right hand side, the first is assumed to be exogenous, and is given by (1) and (3). The second is determined by the requirement that the project should be marginal after tax. For Case F above, this requirement is given by (7), which means that one plus the required expected rate of return before corporate taxes is

$$\frac{E(P)}{\varphi(P)} \cdot \frac{1 - ta - t \frac{c}{1+r}}{1 - t}. \quad (35)$$

The distortion in “one plus the expected rate of return” is the second fraction, which appears in Hall and Jorgenson (1969), p. 395. The distortion is independent of (total and systematic) risk, only a function of tax parameters and the risk free interest rate.

For Case R with an uncertain tax position, the relevant $\varphi(P)Q/I$ ratio for a marginal investment taxed together with inframarginal investment is given in equation (C13) in

Appendix C. One plus the required expected rate of return is

$$\frac{E(P)}{\varphi(P)} \cdot \frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{1 - tN(z_{1D})}. \quad (36)$$

Again the distortion is independent of systematic risk, but now it depends on total risk through the $N(\cdot)$ expressions.

The following proposition summarizes:

Proposition 9: *Under Assumptions 1–3 the required expected return before corporate taxes is given by (35). It is decreasing in a and $c/(1+r)$. It is increasing in the tax rate if $a + c/(1+r) < 1$. The distortion from the tax system does not depend on total or systematic risk. Under Assumptions 1, 2, 4, 6, 7, the required expected return before corporate taxes is given by (36). The distortion from the tax system depends on total risk, but not on systematic risk as long as total risk is unchanged.*

The simplicity of the results may be their most surprising feature.

8 Tax deduction for entry costs?

This section investigates whether there are some conditions for industry equilibrium which would undermine the results from the DRS model. The question arises since the existence of rents will attract entry of new firms. The question has not been raised in the studies cited above. It will be shown that the marginal and average betas differ except under a combination of two conditions: Entry costs exactly outweigh quasi-rents for each firm in operation, and the tax treatment of entry costs is equal to the tax treatment of subsequent investment costs.

Under uncertainty it is natural to assume that access to a unique resource or technology to some extent is the result of a random, risky process. An extreme assumption is that

firms undertake R&D (or exploration for natural resources) with negligible costs and very low success probability. Call this Case A. Under Case A, a small number of firms will have been lucky, and find themselves in the situation described by the model of the previous sections. These firms have only negligible tax shields for R&D costs. The opposite extreme is that all firms which have access to the DRS investment opportunity described here, have paid the same entry cost, and that the after-tax entry cost is equal to the after-tax net value of the investment opportunity (after deduction of $I(1 - ta)$). Call this Case Z. It will be shown below that the important question for the results of this study is to which extent there may be tax shields for the entry costs, in particular in period 1.

The intention is now to find conditions under which the marginal and average betas of equity coincide, to show possible weaknesses of the main results of this paper. The rest of this section thus considers Case Z. It should be clear that there may be intermediate cases between the extremes, in which those firms that have access to the opportunity, have paid some entry cost, but not as much as the net value they obtain from the opportunity. A detailed model of the entry process, its industry equilibrium, and the tax shield consequences of this is omitted here.²⁴

Assumption 8: *An entry cost M is paid for the right to undertake the investment project. This is competitively determined among firms with the same tax position, so that the net value to the firm of paying this entry cost, undertaking the project in optimal scale, and paying taxes, is zero. The sequence of events in period 0 is as follows: (a) The authorities determine the tax system for both periods. (b) The firm pays the entry cost M . (c) The firm determines how much to invest, I .*

In addition to tax deductions defined in Assumption 6, there are tax deductions bM in period 0 and hM in period 1, where b and $h/(1 + r)$ are constants in the interval $[0, 1]$.

To distinguish the expressions from those above, this situation will be called Case G (for Generalized model). The extension of Case R will be developed, while the similar extension of Case F can be found by setting the probabilities (the $N(\cdot)$ expressions) equal to unity.²⁵

Under Assumption 8 there is no economic difference between the two costs, M and I . Only their sum matters to the firm, and the produced quantity might as well have been written as a function of their sum, $M + I$. With no difference between the two, the model would fail to capture the idea of decreasing returns, i.e., the marginal investment project being taxed together with inframarginal investment.

But even under Assumption 8 (i.e., Case Z), the average betas are relevant if there is a difference between the tax treatments of the two costs. The entry cost could be immediately deductible, deductible in the production period, not deductible at all, or some combination of these. The cash flow to equity in period 1 is

$$X_G = Pf(I) - t \cdot \max(Pf(I) - cI - hM, 0). \quad (37)$$

The valuation, as of one period earlier, of a claim to this is

$$\varphi(X_G) = \varphi(P)f(I) - t \left[\varphi(P)f(I)N(z_{1G}) - \frac{cI + hM}{1+r}N(z_{2G}) \right], \quad (38)$$

where

$$z_{1G} = \frac{\ln(\varphi(P)f(I)) - \ln\left(\frac{cI+hM}{1+r}\right)}{\sigma} + \frac{\sigma}{2}, \quad (39)$$

and

$$z_{2G} = z_{1G} - \sigma. \quad (40)$$

Proof of the following proposition is in Appendix D.

Proposition 10: *Under Assumptions 1, 2, 4, 6–8, the beta of equity is given by (41). When there is no deduction for M in period 1 ($h = 0$), then $\beta_{GA} = \beta_{RA}$ (of equation (29)).*

When the two costs M and I are treated equally by the tax system ($a = b, c = h$), then $\beta_{GA} = \beta_{RC}$ (of equation (32)).

The average beta in the general case is given by

$$\beta_{GA} = \frac{1 - ta - \frac{tcN(z_{2G})}{1+r}}{1 - ta - \frac{tcN(z_{2G})}{1+r}(1 - \alpha) + \frac{thN(z_{2G})}{1+r} \cdot \frac{(1-\alpha)[1-t(a + \frac{cN(z_{2G})}{1+r})]}{[1-t(b + \frac{hN(z_{2G})}{1+r})]}} \beta_P, \quad (41)$$

where $z_{2G} =$

$$\frac{1}{\sigma} \left[\ln \left(\frac{1 - ta - \frac{tcN(z_{2G})c}{1+r}}{1 - tN(z_{2G} + \sigma)} (1 + r) \right) - \ln \left(c + h \frac{(1 - \alpha)[1 - t(a + \frac{cN(z_{2G})}{1+r})]}{\alpha[1 - t(b + \frac{hN(z_{2G})}{1+r})]} \right) - \ln(\alpha) \right] - \frac{\sigma}{2}. \quad (42)$$

Only the two special cases will be discussed. The first case implies that b does not matter for the results when $h = 0$, which is due to the fact that the equilibrium $M(1 - tb)$ is determined endogenously. A higher (lower) b will lead to a higher (lower) M , keeping equilibrium $M(1 - tb)$ unaffected, and when $h = 0$, only $M(1 - tb)$ matters, not M separately. For instance, the two subcases ($b = 0, h = 0$) and ($b = 1, h = 0$) give the same beta of equity, β_{RA} , despite the very different tax treatment of M . In these cases with $h = 0$, the difference between marginal and average beta does not go away.

When the two costs are treated equally, the firm's whole activity can be seen as a marginal investment project. In relation to the issues analyzed in this paper, there is nothing which distinguishes this from a case of constant returns to scale, except that the scale of production is determined. The equality of tax treatment, $a = b$ and $c = h$, is an extreme case within the extreme Case Z. Only for this combination of circumstances will the average beta lose its relevance. There are many possible configurations of a, b, c , and h which may be combined with Case Z. Also, Case R above covers at least two interesting possibilities within Case Z, that the entry cost is immediately deductible, and that it is

not deductible at all. More generally, outside of Case Z, it also covers the case of negligible entry costs, which was called Case A.

9 Discussion

There are of course several limitations of the analysis. Assumption 7 on option-like valuation of non-linear cash flows relies on a set of conditions that were not detailed, since they are well known. It should be observed that option-like valuation is not limited to the geometric Brownian motion which is most often used. Other processes have been assumed in some studies, and some of them also allow for analytical solutions. Bradley (1998) considers two alternative stochastic processes for the output price. Likewise, Assumption 1 on the CAPM can be relaxed. The crucial assumption is that the risk measure is linear.

In order to reveal some important analytical results, other simplifying assumptions were also made. Most importantly, only risk free debt was considered, and only in combination with risk free tax deduction.

Among other simplifications, the production function has a constant elasticity. The uncertainty is multiplicative, which may not be necessary for the model to work (cf. Lund 2003a), but for the simplicity of the results, in particular in the case of risky tax shields. The model does not allow for risky inflation, the effect of which would depend on the systematic risk of nominally risk free claims. The source of uncertainty is a single stochastic variable in a single period, and there is no carry-forward or carry-back of losses, all of which exaggerates the risk of the deductions. On the other hand, operating costs in future periods would increase the risk of not being in tax position, thus reducing the risk-adjusted expected values of depreciation tax shields.

The possibility that a multinational may want to change the formal financing of its subsidiaries due to tax changes is neglected here. The subsidiary's borrowing and debt service are likely to be determined by minimization of the total taxes on the global operations of the parent, and possibly by limitations on debt ratios set by authorities in host countries.²⁶ The debt is often formally or de facto guaranteed by the parent. If almost all debt is owed to a related company, bankruptcy and debt capacity cannot be analyzed by the standard methods applicable for stand-alone firms.

Lund (2002a) gives analytical results for a multi-period version of Case F.²⁷ The results are similar to those of section 4 above. As seen from period 0, the effect of investment-related deductions comes through their present value. But the variation over time in this effect is left for future research. The concept of true economic depreciation may be useful to simplify the picture.

In spite of all this, the model should be a step in the direction of more realism, while retaining the possibility of an analytical solution. Hopefully this can be helpful as a reference for numerical examples and empirical studies, when these include the factors which are left out here.

The results are of particular interest under rent taxation with high rates. From a practical viewpoint, serious mistakes will be made when the same discount rate is applied under high and low tax rates (or variations in a versus c). On the theoretical side, there has been a long discussion in the rent taxation literature on which discount rate to use for expected tax shields, or the firm's after-tax net cash flow, cf. the survey by Lund (2009). Under the Resource Rent Tax proposed by Garnaut and Clunies Ross (1975, 1979) the related question is which rate to use for carrying forward negative cash flows. While many have argued that a general number for the firm's cost of capital should be used, one could hope that newer research (relying on Myers (1974)) will lead to a revision of that view.

10 Conclusion

The point of departure of this paper is the literature on tax effects on the after-tax cost of capital, in particular the theoretical models of Galai (1998), Lund (2002a), and Rao and Stevens (2006), but also the previous simulation results in Jacoby and Laughton (1992) and Bradley (1998). The main new result is the tax-induced difference between the marginal and average beta of equity, and its consequences for the weighted average cost of capital. The difference occurs when the tax system allows depreciation or similar deductions in years after investments have been made, and some of the investments are inframarginal. This implies that deductions are proportional to the investment outlay, not to the valuation of the subsequent cash flows. This difference gives rise to the difference in the betas, first demonstrated with risk free tax shields.

The model allows for an analytical approach to the riskiness of depreciation tax shields. The approach exaggerates the riskiness by assuming no loss offset at all. The valuation model originates from the theory of financial options. It reveals that even when the tax shields are risky, they still induce reduced systematic risk of equity. There is still a substantial difference between asset betas and equity betas, even under all-equity financing. The difference between marginal and average equity betas also reappears. Another result is that the rate-of-return shortfall does not affect the ratio of the equity beta to the asset beta, at least not when the production function has constant elasticity.

The inframarginal investments follow from a production function with decreasing returns to scale. A natural question is whether firms have to pay an entry cost to get access to the technology, and how this might affect the results. The analysis shows that only under very strong assumptions would the difference between marginal and average betas of equity disappear. This only happens when all firms that use the technology, have paid

an entry cost equal to the subsequent quasi-rent they earn, in after-tax terms, and at the same time the tax system treats the entry and investment costs equally.

The results on the marginal-average difference imply that some previous studies have failed to identify the cost of capital. Another consequence is that the relationship between observed equity betas and the betas of assets is more complicated than previously believed. When some firms have inframarginal investments and the tax system allows depreciation deductions, not only must equity betas be unlevered, but also “untaxed” and “unaveraged.” Even without inframarginal investments, the need to “untax” is important when required rates of return are being applied under different tax systems (rates and/or deductions).

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Appendix A

Proof of Propositions 1 and 2

Define $\hat{c} \equiv \frac{c}{1+r}$ and $\Delta \equiv 1 - ta - t\hat{c}(1 - \alpha)$. Signs of partial derivatives are needed. These are obtained for β_{FA} to prove Proposition 2. The proof of Proposition 1 follows from setting $\alpha = 1$. The partial derivatives are

$$\frac{\partial \beta_{FA}}{\partial t} = \beta_P \frac{-\hat{c}\alpha}{\Delta^2} < 0, \quad (\text{A1})$$

$$\frac{\partial \beta_{FA}}{\partial a} = \beta_P \frac{-t^2\hat{c}\alpha}{\Delta^2} < 0, \quad (\text{A2})$$

$$\frac{\partial \beta_{FA}}{\partial \hat{c}} = \beta_P \frac{-\alpha t(1 - ta)}{\Delta^2} < 0, \quad (\text{A3})$$

$$\frac{\partial \beta_{FA}}{\partial \alpha} = \beta_P \frac{-t\hat{c}(1 - ta - t\hat{c})}{\Delta^2} < 0, \quad (\text{A4})$$

q.e.d.

Proof of Proposition 3

Define $\bar{m} \equiv E(r_m) - r$. Observe that the expected return on a claim to one unit of the output price satisfies the CAPM: $E(P)/\varphi(P) = 1 + r + \beta_P \bar{m}$.

The maximand based on a risk-adjusted discount rate using the marginal beta is

$$\frac{E(P)f(I)(1 - t) + tcI}{1 + r + \beta_{FM}\bar{m}} - I(1 - ta). \quad (\text{A5})$$

The proposition claims that maximization of this with respect to I gives the same result as (12). The first-order condition is

$$\frac{E(P)f'(I)(1 - t) + tc}{1 + r + \beta_{FM}\bar{m}} = 1 - ta. \quad (\text{A6})$$

Introduce the expression for β_{FM} from (8):

$$E(P)f'(I)(1-t) + tc = (1-ta) \left(1 + r + \beta_P \bar{m} \frac{1-ta-t\hat{c}}{1-ta} \right). \quad (\text{A7})$$

For $E(P)$ introduce the expression $\varphi(P)(1+r+\beta_P \bar{m})$, and find

$$\varphi(P)f'(I)(1-t) = \frac{(1+r)(1-ta-t\hat{c}) + \beta_P \bar{m}(1-ta-t\hat{c})}{1+r+\beta_P \bar{m}} = 1-ta-t\hat{c}, \quad (\text{A8})$$

which is (12). This proves the first part.

Consider now the other part of the proposition, that the average beta can be used, provided that it is considered as a function of I , i.e., $\beta_{FA} = \beta_{FA}(I)$ as defined by (14).

The maximand using the average beta is

$$\frac{E(P)f(I)(1-t) + tcI}{1+r+\beta_{FA}(I)\bar{m}} - I(1-ta). \quad (\text{A9})$$

Introduce $E(P) = \varphi(P)(1+r+\beta_P \bar{m})$ and use equation (14) to rewrite the maximand as

$$\begin{aligned} & \frac{\varphi(P)(1+r+\beta_P \bar{m})f(I)(1-t) + tcI}{1+r + \frac{\varphi(P)f(I)(1-t)}{\varphi(P)f(I)(1-t)+It\hat{c}}\beta_P \bar{m}} - I(1-ta) = \\ & \frac{[\varphi(P)(1+r+\beta_P \bar{m})f(I)(1-t) + tcI][\varphi(P)f(I)(1-t) + It\hat{c}]}{\varphi(P)(1+r+\beta_P \bar{m})f(I)(1-t) + tcI} - I(1-ta), \end{aligned} \quad (\text{A10})$$

which is the same maximand as in (11), q.e.d.

Appendix B

Proof of Propositions 4–5

From equation 20 the first-order condition for a maximum is

$$\varphi(P)f'(I)(1-t) = (1-ta)\Lambda - \frac{tc}{1+r}. \quad (\text{B1})$$

Introduce the analytical production function and rewrite this as

$$\varphi(P)f(I)(1-t) = \frac{I}{\alpha} \left[(1-ta)\Lambda - \frac{tc}{1+r} \right]. \quad (\text{B2})$$

The beta of equity is the value-weighted average of the betas of the elements,

$$\begin{aligned} \beta_{BA} &= \frac{\varphi(P)f(I)(1-t)}{\varphi(P)f(I)(1-t) + I \left[\frac{tc}{1+r} - (\lambda - \eta)(1-ta) \right]} \beta_P \\ &= \frac{(1-ta)\Lambda - \frac{tc}{1+r}}{(1-ta)\Lambda - \frac{tc(1-\alpha)}{1+r} - \alpha(\lambda - \eta)(1-ta)} \beta_P, \end{aligned} \quad (\text{B3})$$

the expression given in the main text.

Observe now that $\partial\Lambda/\partial t = (\eta - 1)r_b t / (1+r) \leq 0$. The tax rate's effect on β_{fA} is given by the partial derivative

$$\frac{\partial\beta_{FA}}{\partial t} = \frac{\alpha\eta \left[(1-ta)^2 \frac{\partial\Lambda}{\partial t} - \frac{c}{1+r} \right]}{\left[(1-ta)\Lambda - \frac{tc(1-\alpha)}{1+r} - \alpha(\lambda - \eta)(1-ta) \right]^2} \beta_P. \quad (\text{B4})$$

This is strictly negative if $\eta < 1$ (which makes $\partial\Lambda/\partial t < 0$) or $c > 0$ or both. This concludes the proof of Proposition 4.

Consider now the application of the WACC, defined as

$$r_w = \eta \{ r + \beta_{BM} [E(r_m) - r] \} + (1-\eta)r_b(1-tg), \quad (\text{B5})$$

as discount rate for the investment decision, the maximization of

$$\frac{E(P)f(I)(1-t) + tcI}{1+r_w} - I(1-ta). \quad (\text{B6})$$

The first-order condition is

$$\frac{E(P)f'(I)(1-t) + tc}{1+r_w} = 1-ta. \quad (\text{B7})$$

Use the notation $\bar{m} \equiv E(r_m) - r$, introduce $E(P) = \varphi(P)(1 + r + \beta_P \bar{m})$, and rewrite as

$$\varphi(P)(1+r+\beta_P \bar{m})f'(I)(1-t)+tc = (1-ta) \left[1 + \eta r + \left(\Lambda - \frac{tc}{(1+r)(1-ta)} \right) \beta_P \bar{m} + (1-\eta)r_b(1-tg) \right] \quad (\text{B8})$$

$$\Leftrightarrow \varphi(P)(1+r+\beta_P \bar{m})f'(I)(1-t) = \Lambda(1-ta)(1+r+\beta_P \bar{m}) - \frac{tc}{1+r}(1+r+\beta_P \bar{m}), \quad (\text{B9})$$

which reduces to

$$\varphi(P)f'(I)(1-t) = \Lambda(1-ta) - \frac{tc}{1+r}, \quad (\text{B10})$$

q.e.d.

Appendix C

Proof of Propositions 6–8

This derivation starts with the average beta in Case R. In Case R the cash flow to equity in period 1 is

$$X_R = Pf(I) - t \cdot \max(Pf(I) - cI, 0). \quad (\text{C1})$$

Under Assumption 7 the valuation, as of one period earlier, of a claim to this is

$$\varphi(X_R) = \varphi(P)f(I) - t \left[\varphi(P)f(I)N(z_{1D}) - \frac{cI}{1+r}N(z_{2D}) \right], \quad (\text{C2})$$

where

$$z_{1D} = \frac{\ln(\varphi(P)f(I)) - \ln\left(\frac{cI}{1+r}\right)}{\sigma} + \frac{\sigma}{2}, \quad (\text{C3})$$

and

$$z_{2D} = z_{1D} - \sigma. \quad (\text{C4})$$

The expression in square brackets in (C2) can be rewritten in terms of the standard Black and Scholes' formula for option pricing as $C(\varphi(P)f(I), cI, 1, r, \sigma)$, so that

$$\varphi(X_R) = \varphi(P)f(I) - tC(\varphi(P)f(I), cI, 1, r, \sigma). \quad (\text{C5})$$

The firm chooses I to maximize $\pi_R(I) \equiv \varphi(X_R) - I(1 - ta)$. The first-order condition is

$$\varphi(P)f'(I) = \frac{\left(1 - ta - tN(z_{2D})\frac{c}{1+r}\right)}{(1 - tN(z_{1D}))}. \quad (\text{C6})$$

Introducing the constant-elasticity production function gives

$$\varphi(P)f(I)(1 - tN(z_{1D})) = \frac{I}{\alpha} \left(1 - ta - tN(z_{2D})\frac{c}{1+r}\right). \quad (\text{C7})$$

The claim is equivalent to holding a portfolio with $f(I)(1 - tN(z_{1D}))$ claims on P , and the rest risk free. The beta is a value-weighted average of the betas of these two elements, i.e.,

$$\beta_{RA} = \frac{\varphi(P)f(I)(1 - tN(z_{1D}))}{\varphi(X_R)}\beta_P. \quad (\text{C8})$$

Here, the subscript RA is introduced to show that this is the average beta in Case R. By introducing the expression for $\varphi(X_R)$ from (C2) and the constant-elasticity production function, this can be simplified as

$$\beta_{RA} = \frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{1 - ta - tN(z_{2D})\frac{c}{1+r}(1 - \alpha)}\beta_P. \quad (\text{C9})$$

It is also possible to express z_{1D} and z_{2D} in terms of exogenous variables, including the elasticity α , avoiding the decision variables of the firm. Plug in from the first-order condition (C7) into (C3)–(C4) to find equation (30) in the main text.

To derive the marginal beta for the same case, consider first the marginal beta derived in Lund (2002a) for the case with an uncertain tax position, equation (24) in that paper. That paper's equation (23) becomes

$$\gamma = \frac{1 - ta - tN(z_{2C})\frac{c}{1+r}}{1 - tN(z_{1C})}, \quad (\text{C10})$$

and the marginal beta can be written

$$\beta_{RC} = \left(1 - ta - tN(z_{2C})\frac{c}{1+r}\right)\beta_P. \quad (\text{C11})$$

The subscript RC (C for CRS) is used here since the case considered in Lund (2002a) did not include the marginal project with some other activity, i.e., as if the case had constant returns to scale.

Again it is possible to express z_{2C} in terms of the exogenous parameters. In this case there is no first-order condition for an interior profit maximum, but the definition of a marginal CRS project, which gives

$$\frac{\varphi(P)Q}{I} = \frac{1 - ta - tN(z_{2C})\frac{c}{1+r}}{1 - tN(z_{1C})}, \quad (\text{C12})$$

cf. equations (5) and (23) in Lund (2002a). This leads to equation (33) in the main text.

What then about the marginal beta for the DRS case? This can be seen as a mixture of the two cases just considered. The marginal beta characterizes a small investment which has a net value of zero. Under imperfect loss offset the value will depend upon the probability of being in tax position. In particular this is crucial in Case R, for which it is assumed that after period one there are no more periods, so that the loss cannot be carried forward (nor backward). The criterion for the project being marginal looks similar to (C12), but in this case the valuation of the option-like cash flow to the marginal project in period 1 is based on the risk-adjusted probabilities $N(z_{1D})$ and $N(z_{2D})$, not $N(z_{1C})$ and $N(z_{2C})$, since they should now reflect the probabilities that the whole DRS project is in tax position at the margin. The project which invests I to yield Q , and which is taxed together with the optimally scaled DRS project, is marginal when

$$\frac{\varphi(P)Q}{I} = \frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{1 - tN(z_{1D})}. \quad (\text{C13})$$

The marginal beta in the DRS case becomes

$$\beta_{RM} = \frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{1 - ta} \beta_P, \quad (\text{C14})$$

with z_{2D} given from (30) in the main text.

Appendix D

Proof of Proposition 10

The firm chooses I to maximize $\pi_G(I) \equiv \varphi(X_G) - I(1 - ta)$. From the first-order condition follows

$$\varphi(P)f(I)(1 - tN(z_{1G})) = \frac{f(I) \left(1 - ta - \frac{tc}{1+r}N(z_{2G})\right)}{f'(I)}. \quad (D1)$$

Introducing the constant-elasticity production function gives

$$\varphi(P)f(I)(1 - tN(z_{1G})) = \frac{I}{\alpha} \left(1 - ta - \frac{tc}{1+r}N(z_{2G})\right). \quad (D2)$$

Equilibrium M is given by

$$\begin{aligned} M(1 - tb) &= \varphi(X_G) - I(1 - ta) \\ &= \frac{I}{\alpha} \left(1 - ta - \frac{tc}{1+r}N(z_{2G})\right) + \frac{tcIN(z_{2G})}{1+r} - I(1 - ta) + \frac{thMN(z_{2G})}{1+r}, \end{aligned} \quad (D3)$$

which can be solved for

$$M = I \frac{(1 - \alpha)[1 - t(a + \frac{cN(z_{2G})}{1+r})]}{\alpha[1 - t(b + \frac{hN(z_{2G})}{1+r})]}. \quad (D4)$$

The ratio of the expressions in square brackets in the numerator and the denominator contains the effect of the different tax treatment (if any) of I and M , respectively, in risk-adjusted expected present value terms.

We can now solve for $\varphi(X_G) =$

$$\frac{I}{\alpha} \left[1 - ta - \frac{tcN(z_{2G})}{1+r}(1 - \alpha) + \frac{thN(z_{2G})}{1+r} \cdot \frac{(1 - \alpha)[1 - t(a + \frac{cN(z_{2G})}{1+r})]}{[1 - t(b + \frac{hN(z_{2G})}{1+r})]} \right]. \quad (D5)$$

This gives the average beta for this case,

$$\beta_{GA} = \frac{1 - ta - \frac{tcN(z_{2G})}{1+r}}{1 - ta - \frac{tcN(z_{2G})}{1+r}(1 - \alpha) + \frac{thN(z_{2G})}{1+r} \cdot \frac{(1 - \alpha)[1 - t(a + \frac{cN(z_{2G})}{1+r})]}{[1 - t(b + \frac{hN(z_{2G})}{1+r})]}} \beta_P, \quad (D6)$$

with $z_{2G} =$

$$\frac{1}{\sigma} \left[\ln \left(\frac{1 - ta - \frac{tN(z_{2G})c}{1+r}}{1 - tN(z_{2G} + \sigma)} (1+r) \right) - \ln \left(c + h \frac{(1-\alpha)[1 - t(a + \frac{cN(z_{2G})}{1+r})]}{\alpha[1 - t(b + \frac{hN(z_{2G})}{1+r})]} \right) - \ln(\alpha) \right] - \frac{\sigma}{2}. \quad (\text{D7})$$

In the first special case, when $h = 0$, the fraction

$$\frac{(1-\alpha)[1 - t(a + \frac{cN(z_{2G})}{1+r})]}{[1 - t(b + \frac{hN(z_{2G})}{1+r})]}, \quad (\text{D8})$$

which appears in both (D6) and (D7), vanishes, since it is multiplied by h . We find $z_{2G} = z_{2D}$ (of equation (30)), and $\beta_{GA} = \beta_{RA}$ (of equation (29)).

In the second special case, when $a = b$ and $c = h$, α vanishes from both (D6) and (D7), since the last two terms in the large square brackets in (D7) are reduced to

$$- \ln \left(c + h \frac{1-\alpha}{\alpha} \right) - \ln(\alpha) = - \ln \frac{c\alpha + c - c\alpha}{\alpha} - \ln(\alpha) = - \ln(c). \quad (\text{D9})$$

Thus we find $z_{2G} = z_{2C}$ (from (33)), and $\beta_{GA} = \beta_{RC}$ (from (32)), q.e.d.

Table I: Beta of equity for the five all-equity subcases, divided by β_P

	$N(z_2)$	α	marginal vs. average	β_i	β_i/β_P
Case F	1	1	marginal	β_{FM}	$\frac{1 - ta - t\frac{c}{1+r}}{1 - ta}$
		$\in (0, 1)$	average	β_{FA}	$\frac{1 - ta - t\frac{c}{1+r}}{1 - ta - t\frac{c}{1+r}(1 - \alpha)}$
Case R	$\in (0, 1)$	1	marginal	β_{RC}	$\frac{1 - ta - tN(z_{2C})\frac{c}{1+r}}{1 - ta}$
		$\in (0, 1)$	marginal	β_{RM}	$\frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{1 - ta}$
		$\in (0, 1)$	average	β_{RA}	$\frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{1 - ta - tN(z_{2D})\frac{c}{1+r}(1 - \alpha)}$
Equations implicitly defining z_{2C} and z_{2D} :					
$z_{2C} = \frac{1}{\sigma} \ln \left(\frac{1 - ta - tN(z_{2C})\frac{c}{1+r}}{[1 - tN(z_{2C} + \sigma)]\frac{c}{1+r}} \right) - \frac{\sigma}{2}$					
$z_{2D} = \frac{1}{\sigma} \ln \left(\frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{\alpha[1 - tN(z_{2D} + \sigma)]\frac{c}{1+r}} \right) - \frac{\sigma}{2}$					

Figure I: β_i/β_P as functions of scale elasticity, α ; $t = 0.35, \sigma = 0.3, c/(1+r) = 1/1.05$

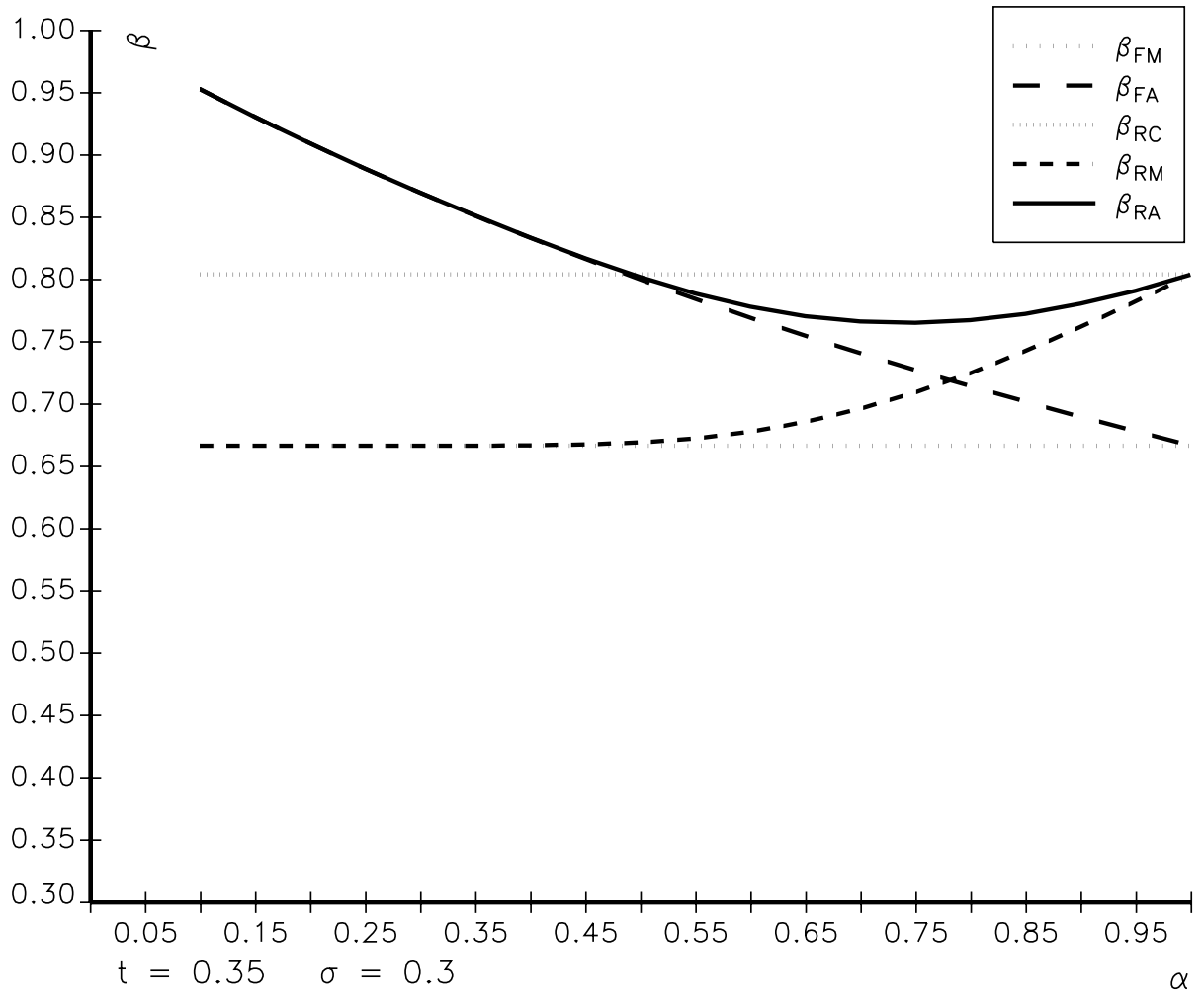


Figure II: β_i/β_P as functions of scale elasticity, α ; varying the tax rate

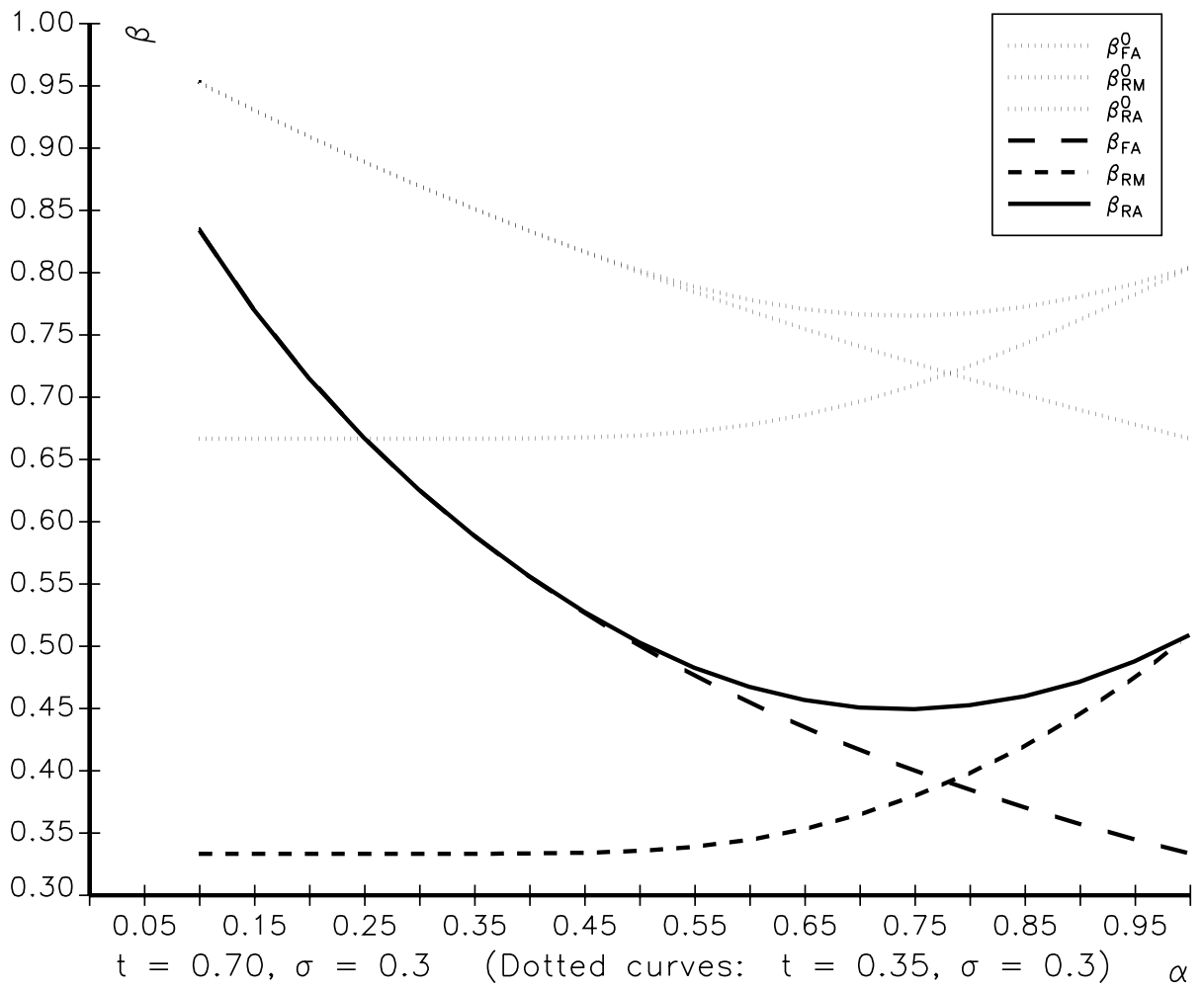
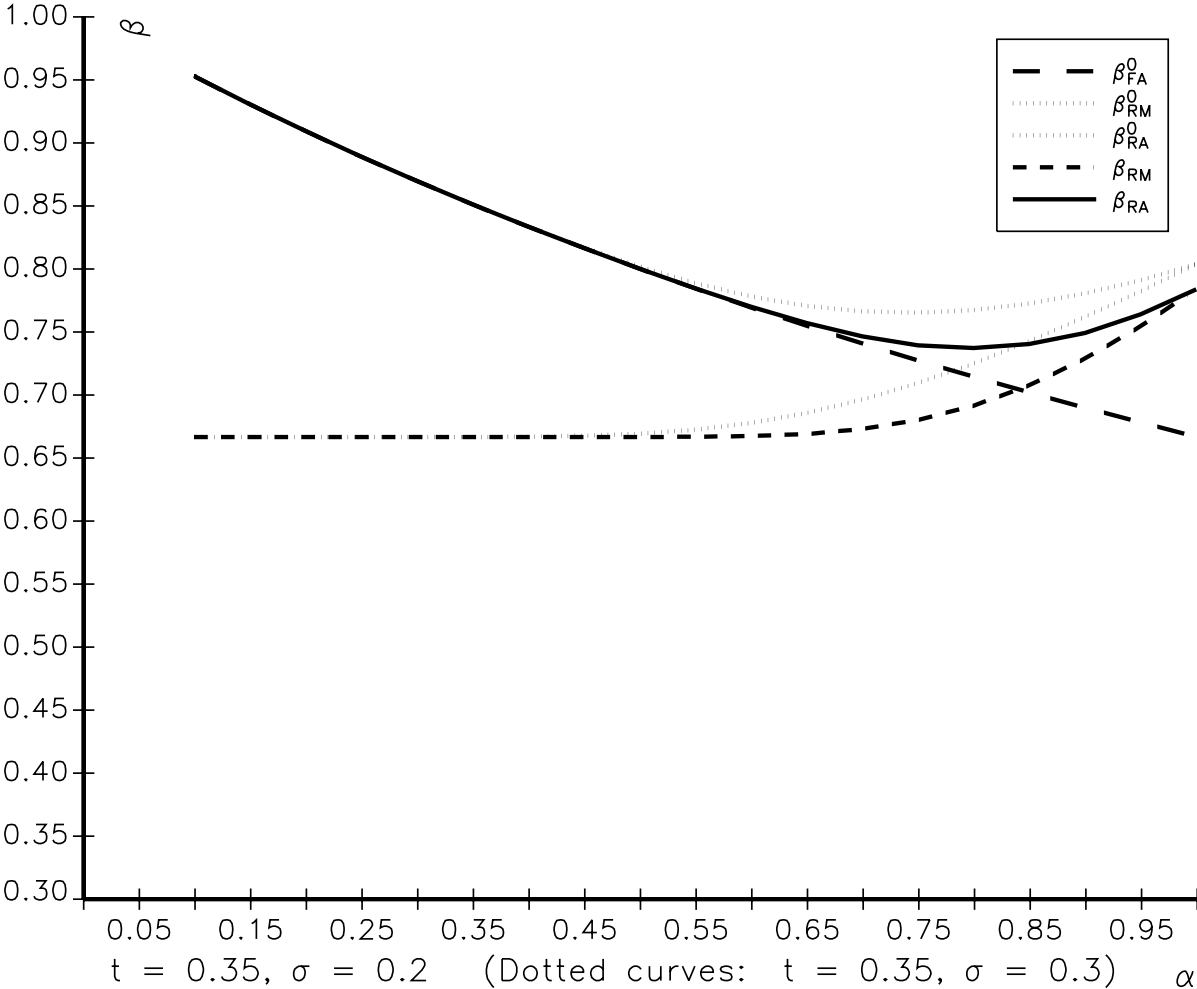


Figure III: β_i/β_P as functions of scale elasticity, α ; varying the volatility



Notes

¹Some have argued that this system is not realistic. In the present paper the system is only used as a pedagogical tool. The relevance of the analysis does not in any way rely on the realism of a Brown tax.

²See Garnaut and Clunies Ross (1975, 1979), Fane (1987), and Bond and Devereux (1995). Such deductions are sometimes known as Allowance for Corporate Equity.

³Without loss offset the government's tax claim is analogous to a call option on the firm's tax base, cf. Ball and Bowers (1983), Green and Talmor (1985), Majd and Myers (1985, 1987).

⁴Brealey, Myers, and Allen (2008) just describe a practice, and do not endorse it. They have another section called "APV for International Investment" (p. 549). But that section does not focus on taxes.

⁵DeAngelo and Masulis (1980) mention several types of non-debt tax shields, such as depreciation allowances, depletion allowances, and investment tax credits. In the present paper only those which are proportional to investment (when the firm is in tax position) are included.

⁶Salahor (1998) has results on these effects in the case of linear taxes, assuming that the firm always will be in tax position and that taxes are proportional.

⁷E.g., they state that "Our interest in this paper is not on the tax shield's risk *per se*" (Rao and Stevens 2006, p. 19f). But, "The sensitivity of interest rate and tax policy

changes on firms' economic balance sheets, and hence investor's wealth can, in principle, be evaluated in our model" (p. 25).

⁸Pitts (1997) obtains analytical results in a similar model, but does not analyze the cost of capital.

⁹The tax system of a small host country would hardly affect the capital market in the home country. But even for tax changes in the U.S. (as host) one may assume that the international capital market is unaffected. Bulow and Summers (1984) indicate that this may be a reasonable assumption (their footnote 3).

¹⁰See Sick (1990), Benninga and Sarig (2003), Cooper and Nyborg (2008). Since the model in sections 3 and 4 has no debt financing, the variable r has only one interpretation, that r which appears as intercept in the CAPM equation, which is also the firm's after-tax discount rate for riskless cash flows.

¹¹In parts of the literature the nominal sum of deductions, here $a + c$, is set to unity (e.g., King 1977, p. 232). In the present paper, a and c are considered as separate, exogenous variables, so that an increase in a is analyzed as if c is kept constant, and vice versa.

¹²This is a special case of Proposition 2 in Lund (2002a).

¹³Lund (2003a) points out that Levy and Arditti (1973) rely on an assumption which may be questionable, that the decision to reinvest indefinitely to compensate for depreciation is made at the beginning of a project.

¹⁴This level is the unspecified $\frac{V}{S}$ in equation (21S') in Galai (1998), who claims (e.g., bottom of p. 144) to find the cost of capital. Rao and Stevens (2006, p. 11) emphasize that their "analysis accommodates both positive and negative NPV firms." They claim

(e.g., middle of p. 2) that the analysis leads to the cost of capital, but the present analysis shows that one must solve for the marginal project. The divergence between the results of Derrig (1994) and the present paper is similar, and is spelt out in Lund (2001).

¹⁵Jacoby and Laughton (1992, p. 44f) and Bradley (1998, p. 69f) do not claim to identify the required expected return. The discount rates they find are based on average betas and are appropriate for finding values of given projects. They do not explicitly recommend them for decision making.

¹⁶See Modigliani and Miller (1963), p. 439.

¹⁷Under some circumstances, not only the cost of debt, but all elements of the WACC will be reduced by a factor of approximately $(1 - t)$: In the cost-of-equity component, both the riskless rate (possibly by one minus the home-country tax rate (see footnote 10)), and β_{BM} (due to depreciation deductions).

¹⁸Shevlin (1990) describes intermediate cases.

¹⁹The present paper improves upon the solution for the case considered in Lund (2002a), by pointing out that the variables z_1 and z_2 , called x_1 and x_2 in equation (19) in that paper, can be rewritten in terms of the exogenous parameters, given that the production function has a constant elasticity. Observe in particular that whereas the option value in general depends on a rate-of-return shortfall, this dependency disappears here, given that the first-order condition of the firm is satisfied.

²⁰As shown in any textbook in finance, $N(z_2)$ is a risk-adjusted probability for the option to be exercised, and $N(z_1)$ multiplies this with a conditional expectation. See, e.g., McDonald (2006).

²¹An alternative would be to rely on an approximate Arbitrage Pricing Theory. Rao and Stevens (2006) rely on this for a related analysis, assuming ad hoc that the approximate valuation equation holds with equality. For the purpose of the present paper, to use the option pricing formula applied here, one must in addition assume that the output price follows a geometric Brownian motion. Leland (1999) points out weaknesses in combining option pricing models with the CAPM. The differences between standard betas and the Bs suggested by Leland are small in relation to the effects pointed out in the present paper.

²²Davis (2002) argues that covariances are likely to change when volatilities of commodity prices change, but does not give any arguments for his assumption that correlations are unchanged. See also the discussion in section 4 of Lund (2005b). Of course, the method used here does not mean that σ could be zero while β is different from zero.

²³See footnote 22.

²⁴The equilibrium will depend on, e.g., whether there is diversification of the entry costs or they are paid by risk averse entrepreneurs, moreover, whether entry is like a lottery with no systematic risk, or a process which in itself has systematic risk, and perhaps decreasing returns to scale.

²⁵The term Case G denotes a tax system, which might have been combined with either Case R or Case F, which distinguish whether tax shields are risky or not. Each of these combinations, GR and GF, might have been considered in combinations with either Case A, Case Z, or an intermediate case. These refer to the amount of entry cost in relation to the subsequent net value for those firms which obtain access to the investment opportunity. Since Case A has negligible entry costs, there is no need to consider it in conjunction with tax system Case G. But intermediate cases might have been considered.

²⁶See Galai (1988), p. 83f., and Pierru and Babusiaux (2008), who find a difference between marginal and average cost of capital, although for reasons different from those of the present paper. The difference they find originates from differences in (statutory) tax rates between jurisdictions.

²⁷Jacoby and Laughton (1992) and Bradley (1998) give numerical results in multi-period models with imperfect loss offset.

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