Granularity Adjustment in Dynamic Multiple Factor Models: Systematic vs Unsystematic Risks

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Abstract

The granularity principle [Gordy (2003)] allows for closed form expressions of the risk measures of a large portfolio at order $1/n$, where $n$ is the portfolio size. The granularity principle yields a decomposition of such risk measures that highlights the different effects of systematic and unsystematic risks. This paper provides the granularity adjustment of the Value-at-Risk (VaR) for both static and dynamic risk factor models. The systematic factor can be multidimensional. The technology is illustrated by several examples, such as the stochastic drift and volatility model, or the model for joint analysis of default and loss given default.

Keywords: Value-at-Risk, Granularity, Large Portfolio, Credit Risk, Systematic Risk, Loss Given Default, Basel 2 Regulation, Credibility Theory.
1 Introduction

Risk measures such as the Value-at-Risk (VaR), the Expected Shortfall (also called Tail VaR) and more generally the Distortion Risk Measures are the basis of the new risk management policies and regulations in both Finance (Basel 2) and Insurance (Solvency 2). These measures are used to define the minimum required capital needed to hedge risky investments; this is the Pillar 1 in Basel 2. They are also used to monitor the risk by means of internal risk models (Pillar 2 in Basel 2).

These risk measures have in particular to be computed for large portfolios of individual contracts, which can be loans, mortgages, life insurance contracts, or Credit Default Swaps (CDS), and for derivative assets written on such large portfolios, such as Mortgage Backed Securities, Collateralized Debt Obligations, derivatives on a CDS index (such as iTraxx, or CDX), Insurance Linked Securities, or longevity bonds. The value of a portfolio risk measure is often difficult to derive even numerically due to

i) the large size of the support portfolio, which can include from about one hundred \(^3\) to several thousands of individual contracts,

ii) the nonlinearity of individual risks, such as default, recovery, claim occurrence, prepayment, surrender, lapse;

iii) the need to take into account the dependence between individual risks, that is induced by the systematic components of these risks.

The granularity principle has been introduced for static single factor models during the discussion on the New Basel Capital Accord [BCBS (2001)], following the contributions by Gordy (2003) and Wilde (2001). The granularity principle allows for closed form expressions of the risk measures for large portfolios at order \(1/n\), where \(n\) denotes the portfolio size. More precisely, any portfolio risk measure can be decomposed as the sum of an asymptotic risk measure corresponding to an infinite portfolio size and \((1/n)\) times an adjustment term. The asymptotic portfolio risk measure, called Cross-Sectional Asymptotic (CSA) risk measure, captures the non diversifiable effect of systematic risks on the portfolio. The adjustment term, called Granularity Adjustment (GA), summarizes the effect of the individual specific risks and their cross-effect with systematic risks, when the portfolio size is large, but finite.

The static risk factor model is introduced in Section 2, whereas the gran-

\(^3\)This corresponds to the number of names included in the iTraxx, or CDX indexes.
ularity adjustment of the VaR is given in Section 3. Section 4 provides the granularity adjustment for a variety of static single and multiple factor risk models. They include models with common stochastic drift and volatility as well as a factor model for the joint analysis of stochastic probability of default (PD) and Expected Loss Given Default (ELGD). The analysis is extended to dynamic risk factor models in Section 5. In the dynamic framework, two granularity adjustments are required. The first GA concerns the conditional VaR with current factor value assumed to be observed. The second GA takes into account the unobservability of the current factor value. This new decomposition relies on recent results on the granularity principle applied to filtering problems [Gagliardini, Gouriéroux (2009)]. The patterns of the granularity adjustment in the credit risk model with stochastic PD and ELGD are illustrated in Section 6. Section 7 concludes. The theoretical derivations of the granularity adjustments are done in the Appendices.

2 Static Risk Factor Model

We first consider the static risk factor model to focus on the individual risks and their dependence structure. We omit the unnecessary time index.

2.1 Homogenous Portfolio

Let us assume that the individual risks (e.g. asset values, or default indicators) depend on some common factors and on individual specific effects:

\[ y_i = c(F, u_i), \quad i = 1, \ldots, n, \]  

(2.1)

where \( y_i \) denotes the individual risk, \( F \) the systematic factor and \( u_i \) the idiosyncratic term. Both \( F \) and \( u_i \) can be multidimensional, whereas \( y_i \) is one-dimensional. Variables \( F \) and \( u_i \) satisfy the following assumptions:

**Distributional Assumptions:** For any portfolio size \( n \):
A.1: \( F \) and \((u_1, \ldots, u_n)\) are independent.
A.2: \( u_1, \ldots, u_n \) are independent and identically distributed.

The portfolio of individual risks is homogenous, since the joint distribution of \((y_1, \ldots, y_n)\) is invariant by permutation of the \( n \) individuals, for any \( n \). This exchangeability property of the individual risks is equivalent to the
fact that variables $y_1, \ldots, y_n$ are independent, identically distributed conditional on some factor $F$ [de Finetti (1931), Hewitt, Savage (1955)]. When the unobservable systematic factor $F$ is integrated out, the individual risks can become dependent.

### 2.2 Examples

The function $c$ in equation (2.1) is given and generally nonlinear. We describe below simple examples of static Risk Factor Model (RFM) (see Section 4 for further examples).

**Example 2.1: Linear Single-Factor Model**

We have:

$$y_i = F + u_i,$$

where the specific error terms $u_i$ are Gaussian $N(0, \sigma^2)$ and the factor $F$ is Gaussian $N(\mu, \eta^2)$. Since $\text{Corr}(y_i, y_j) = \eta^2 / (\eta^2 + \sigma^2)$, for $i \neq j$, the common factor creates the (positive) dependence between individual risks, whenever $\eta^2 \neq 0$. This model has been used rather early in the literature on individual risks. For instance, it is the Buhlmann model considered in actuarial science and is the basis for credibility theory [Buhlmann (1967), Buhlmann, Straub (1970)].

**Example 2.2: The Single Risk Factor Model for Default**

The individual risk is the default indicator, that is $y_i = 1$, if there is a default of individual $i$, and $y_i = 0$, otherwise. This risk variable is given by:

$$y_i = \begin{cases} 
1, & \text{if } F + u_i < 0, \\
0, & \text{otherwise},
\end{cases}$$

where $F \sim N(\mu, \eta^2)$ and $u_i \sim N(0, \sigma^2)$. The quantity $F + u_i$ is often interpreted as a log asset/liability ratio, when $i$ is a company [see e.g. Merton (1974), Vasicek (1991)]. Thus, the company defaults when the asset value becomes smaller than the amount of debt.

The basic specifications in Examples 2.1 and 2.2 can be extended by introducing additional individual heterogeneity, or multiple factors.

**Example 2.3: Model with Stochastic Drift and Volatility**

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The individual risks are such that:

\[ y_i = F_1 + \left( \exp F_2 \right)^{1/2} u_i, \]

where \( F_1 \) (resp. \( F_2 \)) is a common stochastic drift factor (resp. stochastic volatility factor). When \( y_i \) is an asset return, we expect factors \( F_1 \) and \( F_2 \) to be dependent, since the (conditional) expected return generally contains a risk premium.

**Example 2.4: Linear Single Risk Factor Model with Beta Heterogeneity**

This is a linear factor model, in which the individual risks may have different sensitivities (called betas) to the systematic factor. The model is:

\[ y_i = \beta_i F + v_i, \]

where \( u_i = (\beta_i, v_i)' \) is bidimensional. In particular, the betas are assumed unobservable and are included among the idiosyncratic risks. This type of model is the basis of Arbitrage Pricing Theory (APT) [see e.g. Ross (1976), and Chamberlain, Rothschild (1983), in which similar assumptions are introduced on the beta coefficients].

### 3 Granularity Adjustment for Portfolio Risk Measures

#### 3.1 Portfolio Risk

Let us consider an homogenous portfolio including \( n \) individual risks. The total portfolio risk is:

\[ W_n = \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} c(F, u_i). \]  

The total portfolio risk can correspond to a profit and loss (P&L), for instance when \( y_i \) is an asset return and \( W_n/n \) the equally weighted portfolio return. In other cases, it corresponds to a loss and profit (L&P), for instance when \( y_i \)
is a default indicator and \(W_n/n\) the portfolio default frequency\(^4\). As usual, we pass from a P&L to a L&P by a change of sign \(^5\). The quantile of \(W_n\) at a given risk level is used to define a VaR (resp. the opposite of a VaR), if \(W_n\) is a L&P (resp. a P&L).

The distribution of \(W_n\) is generally unknown in closed form due to the risks dependence and the aggregation step. The density of \(W_n\) involves integrals with a large dimension, which can reach \(\dim(F) + n \dim(u) - 1\). Therefore, the quantiles of the distribution of \(W_n\), can also be difficult to compute \(^6\). To address this issue we consider a large portfolio perspective.

### 3.2 Asymptotic Portfolio Risk

The standard limit theorems such as the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT) cannot be applied directly to the sequence \(y_1, \ldots, y_n\) due to the common factors. However, LLN and CLT can be applied conditionally on the factor values, under Assumptions A.1 and A.2. This is the condition of infinitely fine grained portfolio \(^7\) in the terminology of Basel 2 [BCBS (2001)].

Let us denote by

\[
m(F) = E[y_i | F] = E[c(F, u_i) | F],
\]

the conditional individual expected risk, and

\[
\sigma^2(F) = V[y_i | F] = V[c(F, u_i) | F],
\]

\(^4\)The results of the paper are easily extended to obligors with different exposures \(A_i\), say. In this case we have \(W_n = \sum_{i=1}^{n} A_i y_i = \sum_{i=1}^{n} A_i c(F, u_i) = \sum_{i=1}^{n} c^*(F, u_i^*),\) where the idiosyncratic risks \(u_i^* = (u_i, A_i)\) contain the individual shocks \(u_i\) and the individual exposures \(A_i\) [see e.g. Emmer, Tasche (2005) in a particular case].

\(^5\)For instance, this means that asset returns will be replaced by opposite asset returns, that are returns for investors with short positions.

\(^6\)The VaR can often be approximated by simulations, but these simulations can be very time consuming, if the portfolio size is large and the risk level of the VaR is small, especially in dynamic factor models.

\(^7\)Loosely speaking, “... the portfolio is infinitely fine grained, when the largest individual exposure accounts for an infinitely small share of the total portfolio exposure” [Ebert, Lutkebohmert (2009)]. This condition is satisfied under Assumptions A.1 and A.2.
the conditional individual volatility. By applying the CLT conditional on $F$, we get:

$$W_n/n = m(F) + \sigma(F) \frac{X}{\sqrt{n}} + O(1/n), \quad (3.4)$$

where $X$ is a standard Gaussian variable independent of factor $F$. The term at order $O(1/n)$ is zero mean, conditional on $F$, since $W_n/n$ is a conditionally unbiased estimator of $m(F)$.

Expansion (3.4) differs from the expansion associated with the standard CLT. Whereas the first term of the expansion is constant, equal to the unconditional mean in the standard CLT, it is stochastic in expansion (3.4) and linked with the second term of the expansion by means of factor $F$. Moreover, each term in the expansion depends on the factor value, but also on the distribution of idiosyncratic risk by means of functions $m(.)$ and $\sigma(.)$. By considering expansion (3.4), the initial model with $\text{dim}(F) + n \text{dim}(u) - 1$ dimensions of uncertainty is replaced by a 3-dimensional model, with uncertainty summarized by $m(F), \sigma(F)$ and $X$.

### 3.3 Granularity principle

The granularity principle has been introduced for static single risk factor models by Gordy in 2000 for application in Basel 2 [Gordy (2003)]. We extend below this principle to multiple factor models. The granularity principle requires several steps, which are presented below for a loss and profit variable.

**i) A standardized risk measure**

Instead of the VaR of the portfolio risk, which explodes with the portfolio size, it is preferable to consider the VaR per individual risk (asset) included in this portfolio. Since by Assumptions A.1-A.2 the individuals are exchangeable and a quantile function is homothetic, the VaR per individual risk is simply a quantile of $W_n/n$. Intuitively, it is defined to hedge jointly the idiosyncratic risk and a fair part of the systematic risk. The VaR at risk level $\alpha^* = 1 - \alpha$ is denoted by $\text{VaR}_n(\alpha)$ and is defined by the condition:

$$P[W_n/n < \text{VaR}_n(\alpha)] = \alpha, \quad (3.5)$$

where $\alpha$ is a positive number close to 1, typically $\alpha = 95\%, \ 99\%$, which correspond to probabilities of large losses equal to $\alpha^* = 5\%, \ 1\%$, respectively.
ii) The CSA risk measure

Vasicek [Vasicek (1991)] proposed to first consider the limiting case of a portfolio with infinite size. Since

$$\lim_{n \to \infty} \frac{W_n}{n} = m(F), \text{ a.s.},$$

(3.6)

the infinite size portfolio is not riskfree. Indeed, the systematic risk is undiversifiable. We deduce that the CSA risk measure:

$$VaR_\infty(\alpha) = \lim_{n \to \infty} VaR_n(\alpha),$$

(3.7)

is the $\alpha$-quantile associated with the systematic component of the portfolio risk:

$$P[m(F) < VaR_\infty(\alpha)] = \alpha.$$  

(3.8)

The CSA risk measure is suggested in the Internal Ratings Based (IRB) approach of Basel 2 for minimum capital requirements. As seen below, this approach neglects the effect of unsystematic risks.

iii) Granularity Adjustment for the risk measure

The main result in granularity theory applied to risk measures provides the next term in the asymptotic expansion of $VaR_n(\alpha)$ with respect to $n$, for large $n$. It is given below for multiple factor model.

**Proposition 1:** In a static RFM, we have:

$$VaR_n(\alpha) = VaR_\infty(\alpha) + \frac{1}{n} GA(\alpha) + o(1/n),$$

where:

$$GA(\alpha) = -\frac{1}{2} \left\{ \frac{d \log g_\infty}{dm}[VaR_\infty(\alpha)] E[\sigma^2(F)|m(F) = VaR_\infty(\alpha)] + \frac{d}{dm} E[\sigma^2(F)|m(F) = VaR_\infty(\alpha)] \right\}$$

$$= -\frac{1}{2} E \left[ \sigma^2(F)|m(F) = VaR_\infty(\alpha) \right] \left\{ \frac{d \log g_\infty}{dm}[VaR_\infty(\alpha)] + \frac{d}{dm} \log E \left[ \sigma^2(F)|m(F) = VaR_\infty(\alpha) \right] \right\}.$$
and $g_\infty$ [resp. $VaR_\infty(.)$] denotes the probability density function [resp. the quantile function] of the random variable $m(F)$.

**Proof:** See Appendix 1.

This asymptotic expansion of the VaR in Proposition 1 is important for at least four reasons.

(*) The computation of quantities $VaR_\infty(\alpha)$ and $GA(\alpha)$ does not require the evaluation of large dimensional integrals. Indeed, $VaR_\infty(\alpha)$ and $GA(\alpha)$ involve the distribution of transformations $m(F)$ and $\sigma^2(F)$ of the systematic factor only, which are independent of the portfolio size $n$.

(**) The second term in the expansion is of order $1/n$, and not $1/\sqrt{n}$ as might have been expected from the Central Limit Theorem. This implies that the approximation $VaR_\infty(\alpha) + \frac{1}{n}GA(\alpha)$ is likely rather accurate, even for rather small values of $n$ such as $n = 100$.

(***) The expansion is valid for single as well as multiple factor models.

(***) The expansion is easily extended to the other Distortion Risk Measures, which are weighted averages of VaR:

$$DRM_n(H) = \int VaR_n(u)dH(u), \text{ say},$$

where $H$ denotes the distortion measure [Wang (2000)]. The granularity adjustment for the DRM is simply:

$$\frac{1}{n} \int GA(u)dH(u).$$

In particular, the Expected Shortfall corresponds to the distortion measure with cumulative distribution function $H(u;\alpha) = (u - \alpha)^+/(1 - \alpha)$, and the granularity adjustment is

$$\frac{1}{n(1 - \alpha)} \int_\alpha^1 GA(u)du,$$

that is an average of the granularity adjustments for VaR above level $\alpha$. 

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The GA in Proposition 1 depends on the tail magnitude of the systematic risk component \( m(F) \) by means of \( \frac{d \log g_{\infty}}{dm}[VaR_{\infty}(\alpha)] \), which is expected to be negative. The GA depends also on \( \frac{d}{dm} \log E[\sigma^2(F)|m(F) = VaR_{\infty}(\alpha)] \), which is a measure of the reaction of the individual volatility to shocks on the individual drift. When \( y_{i,t} \) is the opposite of an asset return, this reaction function is expected to be nonlinear and increasing for positive values of \( m \), according to the leverage effect interpretation [Black (1976)].

When the tail effect is larger than the leverage effect, the GA is positive, which implies an increase of the required capital compared to the CSA risk measure. In the special case of independent stochastic drift and volatility, the GA reduces to \( GA(\alpha) = -\frac{1}{2} E[\sigma^2(F)|m(F) = VaR_{\infty}(\alpha)] \frac{d \log g_{\infty}}{dm}[VaR_{\infty}(\alpha)] \), which is generally positive. The adjustment involves both the tail of the systematic risk component and the expected conditional variability of the individual risks.

## 4 Examples

The aim of this section is to provide the closed form expressions of the GA for the main models encountered in applications to Finance and Insurance. We first consider single factor models, in which the GA formula is greatly simplified, then models with multiple factor.

### 4.1 Single Risk Factor Model

In a single factor model, the factor can generally be identified with the expected individual risk:

\[
m(F) = F. \tag{4.1}
\]

Then, \( VaR_{\infty}(\alpha) \) is the \( \alpha \)-quantile of factor \( F \) and \( g_{\infty} \) is its density function. The granularity adjustment of the VaR becomes:

\[
GA(\alpha) = -\frac{1}{2} \left\{ \frac{d \log g_{\infty}}{dF}[VaR_{\infty}(\alpha)]\sigma^2[VaR_{\infty}(\alpha)] + \frac{d \sigma^2}{dF}[VaR_{\infty}(\alpha)] \right\}. \tag{4.2}
\]

Example 4.1: Linear Single Risk Factor Model [Gordy (2004)]

In the linear model $y_i = F + u_i$, with $F \sim N(\mu, \eta^2)$ and $u_i \sim N(0, \sigma^2)$, we have $m(F) = F$, $\sigma^2(F) = \sigma^2$, $g_{\infty}(F) = \frac{1}{\eta} \varphi \left( \frac{F - \mu}{\eta} \right)$, and $VaR_{\infty}(\alpha) = \mu + \eta \Phi^{-1}(\alpha)$, where $\varphi$ (resp. $\Phi$) is the density function (resp. cumulative distribution function) of the standard normal distribution. We deduce:

$$GA(\alpha) = -\frac{\sigma^2}{2} \frac{d \log g_{\infty}}{dF} [VaR_{\infty}(\alpha)] = \frac{\sigma^2}{2\eta} \Phi^{-1}(\alpha).$$

In this simple Gaussian framework, the quantile $VaR_n(\alpha)$ is known in closed form. It is easily checked that the above GA corresponds with the first-order term in a Taylor expansion of $VaR_n(\alpha)$ w.r.t. $1/n$ (see Section 5.3). As expected the GA is positive for large $\alpha$ ($\alpha > 0.5$); the GA increases when the idiosyncratic risk increases, that is, when $\sigma^2$ increases. Moreover, the GA is a decreasing function of $\eta$, which means that the adjustment is smaller, when systematic risk increases.

Example 4.2: Static RFM for Default

Let us assume that the individual risks follow independent Bernoulli distributions conditional on factor $F$:

$$y_i \sim B(1, F),$$

where $B(1, p)$ denotes a Bernoulli distribution with probability $p$. This is the well-known model with stochastic probability of default (PD), often called

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reduced form or stochastic intensity model in the Credit Risk literature. In this case \( W_n/n \) is the default frequency in the portfolio. It also corresponds to the portfolio loss given default, if the loans have a unitary nominal and a zero recovery rate. In this model we have \( m(F) = F, \sigma^2(F) = F(1 - F) \), and we deduce:

\[
GA(\alpha) = -\frac{1}{2} \left\{ \frac{d \log g_\infty}{dF} [VAR_\infty(\alpha)] VAR_\infty(\alpha)[1 - VAR_\infty(\alpha)] + 1 - 2 VAR_\infty(\alpha) \right\}.
\] (4.4)

This formula appears for instance in Rau-Bredow (2005).

Different specifications have been considered in the literature for stochastic intensity \( F \).\(^9\) Let us assume that there exists an increasing transformation \( A \), say, from \( (-\infty, +\infty) \) to \( (0,1) \) such that:

\[
A^{-1}(F) \sim N(\mu, \eta^2).
\] (4.5)

We get a probit (resp. logit) normal specification\(^10\), when \( A \) is the cumulative distribution function of the standard normal distribution (resp. the logistic distribution). Let us denote \( a(y) = \frac{dA(y)}{dy} \) the associated derivative. We have:

\[
VAR_\infty(\alpha) = A[\mu + \eta \Phi^{-1}(\alpha)],
\]

\[
\frac{d \log g_\infty}{dF} [VAR_\infty(\alpha)] = -\frac{1}{a[\mu + \eta \Phi^{-1}(\alpha)]} \left( \frac{\Phi^{-1}(\alpha)}{\eta} + \frac{d \log a}{dy} [\mu + \eta \Phi^{-1}(\alpha)] \right).
\] (4.6)

i) Let us first consider the structural Merton (1974) - Vasicek (1991) model [see Example 2.2]. It is known that this model can be written in

\(^9\)Gordy (2004) and Gordy, Lutkebohmert (2007) derive the GA in the CreditRisk+ model [Credit Suisse Financial Products (1997)], which has been the basis for the granularity adjustment proposed in the New Basel Capital Accord [see BCBS (2001, Chapter 8) and Wilde (2001)]. This model has some limitations. First, it assumes that the stochastic PD \( F \) follows a gamma distribution, that admits values of PD larger than 1. Second, it assumes a constant expected loss given default. We present a multi-factor model with stochastic default probability and expected loss given default in Example 4.6 and Section 6.

\(^{10}\)For credit risk, a probit specification is proposed in KMV/Moody’s, whereas a logit specification is used in Credit Portfolio View by Mc Kinsey.
terms of two structural parameters, that are the unconditional probability of default $PD$ and the asset correlation $\rho$, such as:

$$y_i = 1 - \Phi^{-1}(PD) + \sqrt{\rho}F^* + \sqrt{1 - \rho}u^*_i < 0$$

where $F^*$ and $u^*_i$, for $i = 1, \cdots, n$, are independent standard Gaussian variables. The structural factor $F^*$ is distinguished from the reduced form factor $F$ which is the stochastic probability of default. We see that they are related by:

$$\Phi^{-1}(F) = \frac{\Phi^{-1}(PD)}{\sqrt{1 - \rho}} - \frac{\sqrt{\rho}}{\sqrt{1 - \rho}} F^*.$$

We deduce that:

$$\mu = \frac{\Phi^{-1}(PD)}{\sqrt{1 - \rho}}, \quad \eta = \sqrt{\frac{\rho}{1 - \rho}}. \tag{4.7}$$

Thus, from (4.4)-(4.7) we deduce the CSA VaR [Vasicek (1991)]:

$$VaR_\infty(\alpha) = \Phi \left( \frac{\Phi^{-1}(PD) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1 - \rho}} \right), \tag{4.8}$$

and the granularity adjustment:

$$GA(\alpha) = -\frac{1}{2} \left\{ \frac{\Phi^{-1}(PD)}{\sqrt{1 - \rho}} - \frac{1 - 2\rho}{\sqrt{\rho(1 - \rho)}} \Phi^{-1}(\alpha) \right. \left. \phi \left( \frac{\Phi^{-1}(PD) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1 - \rho}} \right) VaR_\infty(\alpha)[1 - VaR_\infty(\alpha)] \right. \\
+1 - 2VaR_\infty(\alpha) \right\}. \tag{4.9}$$

The CSA VaR and the GA depend on both the unconditional probability of default $PD$ and the asset correlation parameter $\rho$.

In Figure 1 we display the CSA and GA VaR as a function of the confidence level for a portfolio of $n = 100$ companies and different combinations of $PD$ and $\rho$, that are $PD = 0.005, 0.05$, and $\rho = 0.12, 0.24$. The two values of $PD$ represent yearly probabilities of default for investment grade and speculative grade companies, respectively. The two values of $\rho$ correspond to the smallest and largest asset correlations, respectively, considered in the Basel 2 regulation [see BCBS(2001)].
For instance, for probability of default $PD = 0.005$ and asset correlation $\rho = 0.12$, the GA CreditVaR of the portfolio default frequency at $\alpha = 95\%$ is about $0.02$. As expected, the CreditVaR is increasing w.r.t. the confidence level $\alpha$, the probability of default $PD$ and the asset correlation $\rho$. The granularity adjustment is positive, that is, it implies a larger minimum required capital. Moreover, the GA is larger for smaller values of the default correlation.

ii) In the logit normal specification, the transformation $A^{-1}(F) = \log[F/(1-F)]$ corresponds to the log of an odd ratio, and the formula for the GA simplifies considerably:

$$VaR_\infty(\alpha) = \frac{1}{1 + \exp[-\mu - \eta\Phi^{-1}(\alpha)]},$$

$$GA(\alpha) = \frac{1}{2\eta}\Phi^{-1}(\alpha).$$

In particular, the GA doesn’t depend on parameter $\mu$.

**Example 4.3: Static RFM for Count Data**

This model is the basic specification for homogenous portfolios of motor insurance contracts. The conditional distribution of the individual risks is $y_t \sim P(F)$ given $F$, where $P(\lambda)$ denotes a Poisson distribution with parameter $\lambda$. Thus, we get a Poisson model with stochastic claim intensity. We have $m(F) = \sigma^2(F) = F$, and we get:

$$GA(\alpha) = -\frac{1}{2} \left\{ \frac{d\log g_\infty}{dF}[VaR_\infty(\alpha)]VaR_\infty(\alpha) + 1 \right\}.$$

Usually, the distribution of $F$ is assumed to be log-normal with parameters $\mu, \eta^2$, that is,

$$\log(F) \sim N(\mu, \eta^2). \quad (4.10)$$

Then:

$$\frac{d\log g_\infty(F)}{dF} = -\frac{1}{F} \left( \frac{\log F - \mu}{\eta^2} + 1 \right), \quad VaR_\infty(\alpha) = \exp[\mu + \eta\Phi^{-1}(\alpha)].$$
and finally:

\[ GA(\alpha) = \frac{1}{2\eta} \Phi^{-1}(\alpha). \] (4.11)

The granularity adjustment is independent of parameter \( \mu \), which is a scale parameter for the stochastic intensity. Since in a Poisson process with intensity \( \lambda \) the number of claims during a period of length \( T \) follows a distribution \( \mathcal{P}(\lambda T) \), this means that the granularity adjustment is the same whether the VaR concerns the number of claims per month, or the number of claims per year. Moreover, the granularity adjustment in the linear Gaussian RFM (Example 4.1), the RFM for default with logit normal specification (Example 4.2), or the RFM for count data with log-normal intensity (Example 4.3), is the same. Indeed, in these models the transformation \( A \) that links the factor \( F \) to a normal variable [see (4.5) and (4.10)] is such that function \( a[A^{-1}(F)] \) is proportional to \( \sigma^2(F) \). In such a case, the GA in (4.2) simplifies to (4.11).

**Example 4.4: Linear Static RFM with Beta Heterogeneity**

Let us consider the model of Example 2.4. We have:

\[ y_i = \beta_i F + v_i, \]

where \( F \sim N(\mu, \eta^2) \), \( v_i \sim N(0, \sigma^2) \), and \( \beta_i \sim N(1, \gamma^2) \), with all these variables independent. Due to a problem of factor identification, the mean of \( \beta_i \) can always be fixed to 1. This will also facilitate the comparison with the model with constant beta of Example 4.1. We get \( \mu(F) = F \), \( \sigma^2(F) = \sigma^2 + \gamma^2 F^2 \), and:

\[ GA(\alpha) = \frac{\sigma^2}{2\eta} \Phi^{-1}(\alpha) + \gamma^2 \left[ VaR_\infty(\alpha)^2 \Phi^{-1}(\alpha) - VaR_\infty(\alpha) \right], \] (4.12)

where \( VaR_\infty(\alpha) = \mu + \eta \Phi^{-1}(\alpha) \). Thus, the CSA risk measure \( VaR_\infty(\alpha) \) is computed in the homogenous model with factor sensitivity \( \beta = 1 \). The granularity adjustment accounts for beta heterogeneity in the portfolio through the variance \( \gamma^2 \) of the heterogeneity distribution. More precisely, the first term in the RHS of (4.12) is the GA already derived in Example 4.1, whereas the second term is specific of the beta heterogeneity.
4.2 Multiple Risk Factor Model

The single risk factor model is too restrictive to capture the complexity of systematic risks. Multiple factors are needed for a joint analysis of stochastic drift and volatility, of default and loss given default, of country and industrial sector specific effects, to monitor the risk of loans with guarantees, when the guarantors themself can default [Ebert, Lutkebohmert (2009)], or to distinguish between trend and cycle effects.

Example 4.5: Stochastic Drift and Volatility Model

i) Let us assume that \( y_i \sim N(F_1, \exp F_2) \), conditional on the bivariate factor \( F = (F_1, F_2)' \), and that

\[
F \sim N \left( \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \eta_1^2 & \rho_{\eta_1\eta_2} \\ \rho_{\eta_1\eta_2} & \eta_2^2 \end{pmatrix} \right).
\]

This type of stochastic volatility model is standard for modelling the dynamic of asset returns, or equivalently of opposite asset returns.

We can introduce the regression equation:

\[
F_2 = \mu_2 + \frac{\rho_{\eta_2}}{\eta_1} (F_1 - \mu_1) + v,
\]

where \( v \) is independent of \( F_1 \), with Gaussian distribution \( N[0, \eta_2^2(1 - \rho^2)] \).

We have \( m(F) = F_1, \sigma^2(F) = \exp(F_2) \), and

\[
E[\sigma^2(F)|m(F)] = E[\exp F_2|F_1]
\]

\[
= E[\exp\{\mu_2 + \frac{\rho_{\eta_2}}{\eta_1} (F_1 - \mu_1) + v\}|F_1]
\]

\[
= \exp[\mu_2 + \frac{\rho_{\eta_2}}{\eta_1} (F_1 - \mu_1)] E[\exp v]
\]

\[
= \exp[\mu_2 + \frac{\rho_{\eta_2}}{\eta_1} (F_1 - \mu_1) + \frac{\eta_2^2(1 - \rho^2)}{2}].
\]

In particular:

\[
\frac{d}{dm} \log E[\sigma^2(F)|m(F)] = \frac{d}{dF_1} \log E[\exp F_2|F_1] = \frac{\rho_{\eta_2}}{\eta_1}.
\]
When $y_i$ is the opposite of an asset return, a positive value $\rho > 0$, i.e. a negative correlation between return and volatility, can represent a leverage effect. From Proposition 1, we deduce that:

$$GA(\alpha) = \frac{1}{2\eta_1}(\Phi^{-1}(\alpha) - \rho \eta_2) \exp[\mu_2 + \eta_2^2/2] \exp[\rho \eta_2 \Phi^{-1}(\alpha) + \rho^2 \eta_2^2/2].$$

The GA of the linear single-factor RFM (see Example 4.1) is obtained when either factor $F_2$ is constant ($\eta_2 = 0$), or factors $F_1$ and $F_2$ are independent ($\rho = 0$), by noting that $E[\exp F_2] = \exp[\mu_2 + \eta_2^2/2]$.

ii) The model above has been written with two factors only, specific of the drift and volatility, respectively. The model encountered in the applications can involve several Gaussian factors $F \sim N(\mu, \Omega)$, say, where $F$ has dimension $d \geq 2$. The individual risks are such that:

$$y_i \sim N[\beta' F, \exp(\gamma' F)],$$

where $\beta$ and $\gamma$ are $d$-dimensional vector of parameters. The granularity adjustment is directly deduced by applying the results of part i) with $F_1 = \beta' F, F_2 = \gamma' F, \mu_2 = \gamma' \mu, \eta_1^2 = \beta' \Omega \beta, \eta_2^2 = \gamma' \Omega \gamma, \rho \eta_1 \eta_2 = \beta' \Omega \gamma$.

Example 4.6: Stochastic Probability of Default and Loss Given Default

Let us consider a portfolio invested in zero-coupon corporate bonds with a same time-to-maturity and identical exposure at default. The individual loss can be written as:

$$y_i = Z_i LGD_i,$$

where $Z_i$ is the default indicator and $LGD_i$ is the loss given default (i.e. one minus the recovery rate). Let $F = (F_1, F_2)'$ denote the vector of factors. We assume that, conditional on factor $F$, the default indicator $Z_i$ and the loss given default $LGD_i$ are independent \(^{11}\), such that $Z_i \sim B(1, F_1)$ and $LGD_i$ admits a beta distribution with parameters $a(F) > 0$ and $b(F) > 0$ that depend on factor $F$. Then:

$$E[LGD_i|F] = \frac{a(F)}{a(F) + b(F)}, \quad V[LGD_i|F] = \frac{E[LGD_i|F] (1 - E[LGD_i|F])}{a(F) + b(F) + 1}.$$

\(^{11}\)But can become dependent, when the unobservable factor is integrated out.
An equivalent parameterization of the beta distribution is in terms of the conditional mean $\mu(F) = a(F)/(a(F) + b(F))$ and the conditional concentration parameter $\gamma(F) = 1/(a(F) + b(F) + 1)$, that are stochastic parameters on $(0, 1)$. The conditional concentration parameter $\gamma(F)$ measures the variability of the conditional distribution of $LGD_i$ given $F$ taking into account that the variance of a random variable on $[0, 1]$ with mean $\mu$, say, is upper bounded by $\mu(1 - \mu)$. When the conditional concentration parameter $\gamma(F)$ approaches 0, the beta distribution degenerates to a point mass; when the conditional concentration parameter $\gamma(F)$ approaches 1 the beta distribution converges to a Bernoulli distribution.

In the sequel, we identify factor $F_2$ with the conditional mean of loss given default, while the concentration parameter $\gamma(F) = \gamma$, with $0 < \gamma < 1$, is assumed constant, independent of the factor. This gives:

$$E[LGD_i|F] = F_2, \quad V[LGD_i|F] = \gamma F_2(1 - F_2).$$

We get a two-factor model, where the stochastic factors are the (conditional) Probability of Default (PD) and the (conditional) Expected Loss Given Default (ELGD).

The conditional mean and volatility of the individual risks are:

$$m(F) = E[LGD_i|F]E[Z_i|F] = F_1 F_2,$$

and:

$$\sigma^2(F) = E[LGD_i^2|F]E[Z_i|F] - E[LGD_i|F]^2 E[Z_i|F]^2$$

$$= \gamma F_2(1 - F_2)F_1 + F_1(1 - F_1)F_2^2,$$

respectively. Thus, the conditional mean is the product of the two factors $F_1$ and $F_2$, while the conditional volatility is a fourth-order polynomial in the factor values. The CSA and GA VaR can be derived by applying Proposition 1 and using equations (4.13) and (4.14). The result is given and discussed in Section 6, where the model is extended to a dynamic framework.

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12 This shows that the mean and the variance cannot be fixed independently for the distribution of a random variable on $[0, 1]$, and explains why we focus on the conditional concentration parameter and not on the conditional variance of $LGD_i$ given $F$.

13 In the standard credit risk models, the LGD is often assumed constant. In such a case the LGD coincides with both its conditional and unconditional expectations. In our framework the LGD is stochastic as well as its conditional expectation.
4.3 Aggregation of Risk Measures

The aggregation of risk measures, in particular of VaR, is a difficult problem. In this section we consider this question in a large portfolio perspective for a single factor model.

Let us index by \( k = 1, \cdots, K \) the different subpopulations, with respective weights \( \pi_k, k = 1, \cdots, K \). Within subpopulation \( k \), the conditional mean and variance given factor \( F \) are denoted \( m_k(F) \) and \( \sigma_k^2(F) \), respectively. At population level, the conditional mean is:

\[
\bar{m}(F) = \sum_{k=1}^{K} \pi_k m_k(F),
\]

by the iterated expectation theorem, and the conditional variance is given by:

\[
\bar{\sigma}^2(F) = \sum_{k=1}^{K} \sigma_k^2(F) + \sum_{k=1}^{K} \pi_k [m_k(F) - \bar{m}(F)]^2,
\]

by the variance decomposition equation. We denote by \( Q_\infty, q_\infty = \frac{dQ_\infty}{dF} \), \( G_\infty \) and \( g_\infty \) the quantile, quantile density, cdf and pdf of the distribution of factor \( F \), respectively.

i) Aggregation of CSA risk measures

We assume:

A.3: The conditional means \( m_k(F) \) are increasing functions of \( F \), for any \( k = 1, \cdots, K \).

Under Assumption A.3, the systematic risk components \( m_k(F), k = 1, \cdots, K \) are co-monotonic. Therefore, we have:

\[
P[\bar{m}(F) < \overline{\text{VaR}}_\infty(\alpha)] = \alpha \iff P[F < \bar{m}^{-1}(\overline{\text{VaR}}_\infty(\alpha))] = \alpha,
\]

and we deduce that the VaR at population level is:

\[
\overline{\text{VaR}}_\infty(\alpha) = \bar{m}[Q_\infty(\alpha)] = \sum_{k=1}^{K} \pi_k m_k[Q_\infty(\alpha)] = \sum_{k=1}^{K} \pi_k \text{VaR}_k(\alpha), \quad (4.15)
\]
by definition of the disaggregated VaR. Thus, we have the perfect aggregation formula for the CSA risk measure, that is, the aggregate VaR is obtained by summing the disaggregated VaR with the subpopulation weights.

**ii) Aggregation of Granularity Adjustments**

We have the following Proposition:

**Proposition 2:** The aggregate granularity adjustment is given by:

**Proof:** See Appendix 3. [TO BE CONTINUED]

### 5 Dynamic Risk Factor Model (DRFM)

The static risk factor model implicitly assumes that the past observations are not informative to predict the future risk. In dynamic factor models, the VaR becomes a function of the available information. This conditional VaR has to account for the unobservability of the current and lagged factor values. We show in this section that factor unobservability implies an additional GA for the VaR. Despite this further layer of complexity in dynamic models, the granularity principle becomes even more useful compared to the static framework. Indeed, the conditional cdf of the portfolio value at date $t$ involves an integral that can reach dimension $(t + 1)\dim(F) + n \dim(u) - 1$, which now depends on $t$, due to the integration w.r.t. the factor path.

#### 5.1 The Model

Dynamic features can easily be introduced in the following way:

1) We still assume a static relationship between the individual risks and the systematic factors. This relationship is given by the static measurement equations:

$$y_{it} = c(F_t, u_{it}),  \quad (5.1)$$

where the idiosyncratic risks $(u_{i,t})$ are independent, identically distributed across individuals and dates, and independent of the factor process $(F_t)$. 


ii) Then, we allow for factor dynamic. The factor process \((F_t)\) is Markov with transition pdf \(g(f_t|f_{t-1})\), say. Thus, all the dynamics of individual risks pass by means of the factor dynamic.

Let us now consider the future portfolio risk per individual asset defined by \(W_{n,t+1}/n = \frac{1}{n} \sum_{i=1}^{n} y_{i,t+1}\). The (conditional) VaR is defined by the equation:

\[
P[W_{n,t+1}/n < \text{VaR}_{n,t}(\alpha)|I_{n,t}] = \alpha,
\]

where the available information \(I_{n,t}\) includes all current and past individual risks \(y_{i,t}, y_{i,t-1}, \ldots\), for \(i = 1, \ldots, n\), but not the current and past factor values. The conditional quantile \(\text{VaR}_{n,t}(\alpha)\) depends on the date \(t\) through the information \(I_{n,t}\).

### 5.2 Granularity Adjustment

i) Asymptotic expansion of the portfolio risk

Let us first perform an asymptotic expansion of the portfolio risk. By the cross-sectional CLT applied conditional on the factor path, we have:

\[
W_{n,t+1}/n = m(F_{t+1}) + \sigma(F_{t+1}) \frac{X_{t+1}}{\sqrt{n}} + O(1/n),
\]

where the variable \(X_{t+1}\) is standard normal, independent of the factor process, and \(O(1/n)\) denotes a term of order \(1/n\), which is zero-mean conditional on \(F_{t+1}, F_{t}, \ldots\). The functions \(m(.)\) and \(\sigma^2(.)\) are defined analogously as in (3.2)-(3.3), and depend on \(F_{t+1}\) only by the static measurement equations (5.1).

In order to compute the conditional cdf of \(W_{n,t+1}/n\), it is useful to reintroduce the current factor value in the conditioning set through the law of
iterated expectation. We have:

\[
P[W_{n,t+1}/n < y|I_{n,t}] = E[P(W_{n,t+1}/n < y|F_t, I_{n,t})|I_{n,t}]
\]

\[
= E[P(W_{n,t+1}/n < y|F_t)|I_{n,t}]
\]

\[
= E[P(m(F_{t+1}) + \sigma(F_{t+1})\frac{X_{t+1}}{\sqrt{n}} + O(1/n) < y|F_t)|I_{n,t}]
\]

\[
= E[a(y, \frac{X_{t+1}}{\sqrt{n}} + O(1/n); F_t)|I_{n,t}],
\]

where function \(a\) is defined by:

\[
a(y, \varepsilon; f_t) = P[m(F_{t+1}) + \sigma(F_{t+1})\varepsilon < y|F_t = f_t].
\]

(5.4)

ii) Cross-sectional approximation of the factor

Function \(a\) depends on the unobserved factor value \(F_t = f_t\), and we have first to explain how this value can be approximated from observed individual variables. For this purpose, let us denote by \(h(y_{i,t}|f_t)\) the conditional density of \(y_{i,t}\) given \(F_t = f_t\), deduced from model (5.1), and define the cross-sectional maximum likelihood approximation of \(f_t\) given by:

\[
\hat{f}_{n,t} = \arg \max_{f_t} \sum_{i=1}^{n} \log h(y_{i,t}|f_t).
\]

(5.5)

Hence, the factor value \(f_t\) is treated as an unknown parameter in the cross-sectional conditional model at date \(t\) and is approximated by the maximum likelihood principle. Approximation \(\hat{f}_{nt}\) is a function of the current individual observations and hence of the available information \(I_{n,t}\).

iii) Granularity Adjustment for factor prediction

It might seem natural to replace the unobserved factor value \(f_t\) by its cross-sectional approximation \(\hat{f}_{n,t}\) in the expression of function \(a\), and then to use the GA of the static model in Proposition 1, for the distribution of \(F_{t+1}\) given \(F_t = \hat{f}_{nt}\). However, replacing \(f_t\) by \(\hat{f}_{n,t}\) implies an approximation
error. It has been proved that this error is of order $1/n$, that is, the same order expected for the GA. More precisely, we have the following result which is given in the single factor framework for expository purpose [Gagliardini, Gouriéroux (2009), Corollary 5.3]:

**Proposition 3:** Let us consider a dynamic single factor model. For a large homogenous portfolio, the conditional distribution of $F_t$ given $I_{n,t}$ is approximately normal at order $1/n$:

$$N\left(\hat{f}_{n,t} + \frac{1}{n}\mu_{n,t}, \frac{1}{n}J_{n,t}^{-1}\right),$$

where:

$$\mu_{n,t} = J_{n,t}^{-1}\frac{\partial \log g}{\partial f_t}(\hat{f}_{n,t} | \hat{f}_{n,t-1}) + \frac{1}{2} J_{n,t}^{-2} K_{n,t},$$

$$J_{n,t} = -\frac{1}{n} \sum_{i=1}^{n} \frac{\partial^2 \log h}{\partial f_t^2}(y_{i,t} | \hat{f}_{n,t}),$$

$$K_{n,t} = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial^3 \log h}{\partial f_t^3}(y_{i,t} | \hat{f}_{n,t}).$$

Proposition 3 gives an approximation of the filtering distribution of factor $F_t$ given the information $I_{n,t}$. This approximation involves four summary statistics, which are the cross-sectional maximum likelihood approximations $\hat{f}_{n,t}$ and $\hat{f}_{n,t-1}$, the Fisher information $J_{n,t}$ for approximating the factor in the cross-section at date $t$, and the statistic $K_{n,t}$ in the bias adjustment.

**iii) Expansion of the cdf of portfolio risk**

From equation (5.3) and Proposition 3, the conditional cdf of the portfolio risk can be written as:

$$P[W_{n,t+1/n} < y | I_{n,t}] = E\left[a\left(y, \frac{X_{t+1}}{\sqrt{n}} + O(1/n); \hat{f}_{n,t} + \frac{1}{n}\mu_{n,t} + \frac{1}{\sqrt{n}} J_{n,t}^{-1/2} X_t^* | I_{n,t}\right) - a(1/n),
$$
where $X_t^*$ is a standard Gaussian variable independent of $X_{t+1}$ and $O(1/n)$, of the factor path and of the available information\(^{14}\).

Then, we can expand the expression above with respect to $n$, up to order $1/n$. By noting that $\hat{f}_{n,t}$, $\mu_{n,t}$, $J_{n,t}$ are functions of the available information and that $E[X_{t+1}] = E[X_t^*] = E[O(1/n)] = 0$, $E[X_{t+1}X_t^*] = 0$, $E[X_{t+1}^2] = E[(X_t^*)^2] = 1$, we get:

$$P[W_{n,t+1}/n < y | I_{n,t}] = a(y, 0; \hat{f}_{n,t}) + \frac{1}{n} \frac{\partial a(y, 0; \hat{f}_{n,t})}{\partial f_t} \mu_{n,t} + \frac{1}{2n} \left[ J_{n,t}^{-1} \frac{\partial^2 a(y, 0; \hat{f}_{n,t})}{\partial f_t^2} + \frac{\partial^2 a(y, 0; \hat{f}_{n,t})}{\partial \varepsilon^2} \right] + o(1/n).$$

The CSA conditional cdf of the portfolio risk is $a(y, 0; \hat{f}_{n,t})$, where:

$$a(y, 0; f_t) = P[m(F_{t+1}) < y | F_t = f_t] \quad (5.6)$$

It corresponds to the conditional cdf of $m(F_{t+1})$ given $F_t = f_t$, where the unobservable factor value $f_t$ is replaced by its cross-sectional approximation $\hat{f}_{n,t}$. The GA for the cdf is the sum of two components corresponding to:

(*) The granularity adjustment for the conditional cdf with known $F_t$ equal to $\hat{f}_{n,t}$, that is,

$$\frac{1}{2n} \frac{\partial^2 a(y, 0; \hat{f}_{n,t})}{\partial \varepsilon^2}. \quad (5.7)$$

The second-order derivative of function $a(y, \varepsilon; f_t)$ w.r.t. $\varepsilon$ at $\varepsilon = 0$ can be computed by using Lemma a.1 in Appendix 1, which yields:

$$\frac{\partial^2 a(y, 0; f_t)}{\partial \varepsilon^2} = \frac{d}{dy} \left\{ g_{\infty}(y; f_t) E[\sigma^2(F_{t+1}) | m(F_{t+1}) = y, F_t = f_t] \right\},$$

where $g_{\infty}(y; f_t)$ denotes the pdf of $m(F_{t+1})$ conditional on $F_t = f_t$.

(**) The granularity adjustment for filtering, that is,

$$\frac{\partial a(y, 0; \hat{f}_{n,t})}{\partial f_t} \mu_{n,t} + \frac{1}{2} J_{n,t}^{-1} \frac{\partial^2 a(y, 0; \hat{f}_{n,t})}{\partial f_t^2}. \quad (5.8)$$

\(^{14}\)The independence between $X_t^*$ and $X_{t+1}$ is due to the fact that $X_t^*$ simply represents the numerical approximation of the filtering distribution of $F_t$ given $I_{n,t}$ and is not related to the stochastic features of the observations at $t + 1$ [see Gagliardini, Gouriéroux (2009)].
It involves the first- and second-order derivatives of the CSA cdf w.r.t. the conditioning factor value.

Due to the independence between variables $X_{t+1}$ and $X^*_t$, there is no cross GA.

iv) Granularity Adjustment for the VaR

The CSA cdf is used to define the CSA risk measure $VaR_\infty (\alpha; \hat{f}_{n,t})$ through the condition:

$$P \left[ m(F_{t+1}) < VaR_\infty (\alpha; \hat{f}_{n,t}) | F_t = \hat{f}_{n,t} \right] = \alpha.$$

The CSA VaR depends on the current information through the cross-sectional approximation of the factor value $\hat{f}_{n,t}$ only. The GA for the (conditional) VaR is directly deduced from the GA of the (conditional) cdf by applying the Bahadur’s expansion [Bahadur (1966); see Lemma a.3 in Appendix 1]. We get the next Proposition:

**Proposition 4:** In a dynamic RFM the (conditional) VaR is such that:

$$VaR_{n,t}(\alpha) = VaR_\infty (\alpha; \hat{f}_{n,t}) + \frac{1}{n} \left[ GA_{risk}(\alpha) + GA_{filt}(\alpha) \right] + o(1/n),$$

where:

$$GA_{risk}(\alpha) = -\frac{1}{2} \left\{ \frac{\partial \log g_\infty (y; \hat{f}_{n,t})}{\partial y} E[\sigma^2(F_{t+1}) | m(F_{t+1}) = y, F_t = \hat{f}_{n,t}] + \frac{\partial}{\partial y} E[\sigma^2(F_{t+1}) | m(F_{t+1}) = y, F_t = \hat{f}_{n,t}] \right\}_{y = VaR_\infty (\alpha; \hat{f}_{n,t})},$$

and:

$$GA_{filt}(\alpha) = -\frac{1}{g_\infty [VaR_\infty (\alpha; \hat{f}_{n,t}); \hat{f}_{n,t}]} \left\{ \frac{\partial a[VaR_\infty (\alpha, \hat{f}_{n,t}), 0; \hat{f}_{n,t}]}{\partial \hat{f}_t} \right\}_{\hat{f}_{n,t}},$$

and where $g_\infty (\cdot; f_t)$ [resp. $a(\cdot, 0; f_t)$ and $VaR_\infty (\cdot; f_t)$] denotes the pdf [resp. the cdf and quantile] of $m(F_{t+1})$ conditional on $F_t = f_t$. 

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Thus, the GA for the conditional VaR is the sum of two components. The first one \( GA_{\text{risk}}(\alpha) \) is the analogue of the GA in the static factor model (see Proposition 1). However, the distribution of \( m(F_{t+1}) \) and \( \sigma^2(F_{t+1}) \) is now conditional on \( F_t = f_t \), and the unobservable factor value \( f_t \) is replaced by its cross-sectional approximation \( \hat{f}_{n,t} \). The second component \( GA_{\text{filt}}(\alpha) \) is due to the filtering of the unobservable factor value, and involves first- and second-order derivatives of the CSA cdf w.r.t. the conditioning factor value.

5.3 Linear RFM with AR(1) factor

As an illustration, let us consider the model given by:

\[
y_{i,t} = F_t + u_{i,t}, \quad i = 1, \ldots, n,
\]

where:

\[
F_t = \mu + \rho(F_{t-1} - \mu) + v_t,
\]

and \( u_{i,t}, v_t \) are independent, with \( u_{it} \sim \text{IN}(0, \sigma^2), v_t \sim \text{IN}(0, \eta^2) \). In this Gaussian framework the (conditional) VaR can be computed explicitly, which allows for a comparison with the granularity approximation in order to assess the accuracy of the two GA components and their (relative) magnitude.

i) The static case

Let us first set \( \rho = 0 \), which corresponds to the static case (see Example 4.1). The distribution of:

\[
W_{n,t+1}/n = \bar{y}_{n,t+1} = F_{t+1} + \bar{u}_{n,t+1} = \mu + v_{t+1} + \bar{u}_{n,t+1},
\]

is Gaussian with mean \( \mu \) and variance \( \eta^2 + \frac{\sigma^2}{n} \), where \( \bar{y}_{n,t+1} \) denotes the cross-sectional average of the individual risks at date \( t + 1 \) and similarly for \( \bar{u}_{n,t+1} \). We deduce the static VaR given by:

\[
VaR_n(\alpha) = \mu + \sqrt{\eta^2 + \frac{\sigma^2}{n}} \Phi^{-1}(\alpha).
\]

By expanding the square root term at order \( 1/n \), we get:

\[
VaR_n(\alpha) = \mu + \eta \Phi^{-1}(\alpha) + \frac{1}{n} \frac{\sigma^2}{2\eta} \Phi^{-1}(\alpha) + o(1/n),
\]
where the first term is the VaR associated with factor $F_t$ and the second term is the GA derived in Example 4.1.

ii) The dynamic case

Let us now consider the general case. The individual observations can be summarized by their cross-sectional averages\(^{15}\) and we have:

\[
\begin{align*}
\bar{y}_{n,t+1} &= F_{t+1} + \bar{u}_{n,t+1}, \\
F_{t+1} &= \mu + \rho(F_t - \mu) + v_{t+1}.
\end{align*}
\]

(5.9)

This implies:

\[
\bar{y}_{n,t+1} = \mu + \frac{1}{1 - \rho L} v_{t+1} + \bar{u}_{n,t+1}
\]

\[
= \mu + \frac{1}{1 - \rho L} (v_{t+1} + \bar{u}_{n,t+1} - \rho \bar{u}_{n,t}),
\]

where $L$ denotes the lag operator. The process $Z_{t+1} = v_{t+1} + \bar{u}_{n,t+1} - \rho \bar{u}_{n,t}$ is a Gaussian MA(1) process that can be written as $Z_{t+1} = \varepsilon_{t+1} - \theta_n \varepsilon_t$, where the variables $\varepsilon_t$ are $IN(0, \gamma_n^2)$, say, and $|\theta_n| < 1$. The new parameters $\theta_n$ and $\gamma_n$ are deduced from the expressions of the variance and autocovariance at lag 1 of process $(Z_t)$. They satisfy:

\[
\begin{align*}
\eta^2 + \frac{\sigma^2}{n}(1 + \rho^2) &= \gamma_n^2(1 + \theta_n^2), \\
\frac{\rho \sigma^2}{n} &= \theta_n \gamma_n^2.
\end{align*}
\]

(5.10)

Hence:

\[
\theta_n = \frac{b_n - \sqrt{b_n^2 - 4 \rho^2}}{2 \rho}, \quad \gamma_n^2 = \frac{\rho \sigma^2}{n \theta_n},
\]

(5.11)

where $b_n = 1 + \rho^2 + n \frac{\eta^2}{\sigma^2}$ and we have selected the root $\theta_n$ such that $|\theta_n| < 1$. Therefore, variable $\bar{y}_{n,t+1}$ follows a Gaussian ARMA(1,1) process, and we can

\(^{15}\)By writing the likelihood of the model, it is seen that the cross-sectional averages are sufficient statistics.
write:

$$\bar{y}_{n,t+1} = \mu + \frac{1 - \theta_n L}{1 - \rho L} \varepsilon_{t+1}$$

$$\iff \bar{y}_{n,t+1} = \mu + \left(1 - \frac{1 - \rho L}{1 - \theta_n L}\right) (\bar{y}_{n,t+1} - \mu) + \varepsilon_{t+1}$$

$$\iff \bar{y}_{n,t+1} = \mu + \frac{\rho - \theta_n}{1 - \theta_n L} (\bar{y}_{n,t} - \mu) + \varepsilon_{t+1}$$

$$\iff \bar{y}_{n,t+1} = \mu + (\rho - \theta_n) \sum_{j=0}^{\infty} \theta_n^j (\bar{y}_{n,t-j} - \mu) + \varepsilon_{t+1}.$$ 

We deduce that the conditional distribution of $W_{n,t+1}/n = \bar{y}_{n,t+1}$ given $I_{n,t}$ is Gaussian with mean $\mu + (\rho - \theta_n) \sum_{j=0}^{\infty} \theta_n^j (\bar{y}_{n,t-j} - \mu)$ and variance $\gamma_n^2$.

**Proposition 5:** In the linear RFM with AR(1) Gaussian factor, the conditional VaR is given by:

$$\text{VaR}_{n,t}(\alpha) = \mu + (\rho - \theta_n) \sum_{j=0}^{\infty} \theta_n^j (\bar{y}_{n,t-j} - \mu) + \gamma_n \Phi^{-1}(\alpha),$$

where $\theta_n$ and $\gamma_n$ are given in (5.11).

Thus, the conditional VaR depends on the information $I_{n,t}$ through a weighted sum of current and lagged cross-sectional individual risks averages. The weights decay geometrically with the lag as powers of parameter $\theta_n$. Moreover, the information $I_{n,t}$ impacts the VaR uniformly in the risk level $\alpha$.

Let us now derive the expansion of $\text{VaR}_{n,t}(\alpha)$ at order $1/n$ for large $n$. From (5.11), the expansions of parameters $\theta_n$ and $\gamma_n$ are:

$$\theta_n = \frac{\rho \sigma^2}{\eta^2 n} + o(1/n), \quad \gamma_n = \eta + \frac{\sigma^2}{2n \eta} (1 + \rho^2) + o(1/n).$$

As $n \to \infty$, the MA parameter $\theta_n$ converges to zero and the variance of the shocks $\gamma_n^2$ converges to $\eta^2$. Hence, the ARMA(1,1) process of the cross-sectional averages $\bar{y}_{n,t}$ approaches the AR(1) factor process ($F_t$) as expected.
By plugging the expansions for $\theta_n$ and $\gamma_n$ into the expression of $VaR_{n,t}(\alpha)$ in Proposition 5, we get:

\[
VaR_{n,t}(\alpha) = \mu + \rho(\bar{y}_{n,t} - \mu) + \eta \Phi^{-1}(\alpha) + \frac{1}{n} \left\{ \frac{\sigma^2}{2\eta} (1 + \rho^2) \Phi^{-1}(\alpha) - \frac{\rho \sigma^2}{\eta^2} [\bar{y}_{n,t} - \rho(\bar{y}_{n,t-1} - \mu)] \right\} + o(1/n).
\]

The first row on the RHS provides the CSA VaR:

\[
VaR_\infty(\alpha; \hat{f}_{n,t}) = \mu + \rho(\bar{y}_{n,t} - \mu) + \eta \Phi^{-1}(\alpha).
\]

which depends on the information through the cross-sectional maximum likelihood approximation of the factor $\hat{f}_{n,t} = \bar{y}_{n,t}$. The CSA VaR is the quantile of the normal distribution with mean $\mu + \rho(\bar{y}_{n,t} - \mu)$ and variance $\eta^2$, that is the conditional distribution of $F_{t+1}$ given $F_t = \bar{y}_{n,t}$. The GA involves the information $I_{n,t}$ through the current and lagged cross-sectional averages $\bar{y}_{n,t}$ and $\bar{y}_{n,t-1}$. The other lagged values $\bar{y}_{n,t-j}$ for $j \geq 2$ are irrelevant at order $o(1/n)$.

Let us now identify the risk and filtering GA components. We have:

\[
a(y, 0; f_t) = P(F_{t+1} < y | F_t = f_t) = \Phi \left( \frac{y - \mu - \rho(f_t - \mu)}{\eta} \right).
\]

We deduce:

\[
g_\infty(y; f_t) = \frac{\partial a(y, 0; f_t)}{\partial y} = \frac{1}{\eta} \varphi \left( \frac{y - \mu - \rho(f_t - \mu)}{\eta} \right),
\]

and:

\[
\frac{\partial a(y, 0; f_t)}{\partial f_t} = -\frac{\rho}{\eta} \varphi \left( \frac{y - \mu - \rho(f_t - \mu)}{\eta} \right),
\]

\[
\frac{\partial^2 a(y, 0; f_t)}{\partial f_t^2} = \frac{\rho^2}{\eta^2} \varphi \left( \frac{y - \mu - \rho(f_t - \mu)}{\eta} \right) \varphi \left( \frac{y - \mu - \rho(f_t - \mu)}{\eta} \right).
\]

Moreover, the statistics involved in the approximate filtering distribution of $F_t$ given $I_{n,t}$ (see Proposition 3) are $\mu_{n,t} = -\frac{\sigma^2}{\eta^2} [(\bar{y}_{n,t} - \mu) - \rho(\bar{y}_{n,t-1} - \mu)]$, $J_{n,t} = 1/\sigma^2$ and $K_{n,t} = 0$. From Proposition 3 and equation (5.12), we get:
\begin{align*}
GA_{\text{risk}}(\alpha) &= \frac{\sigma^2}{2\eta} \Phi^{-1}(\alpha), \\
GA_{\text{filt}}(\alpha) &= \frac{\sigma^2 \rho^2}{2\eta} \Phi^{-1}(\alpha) - \frac{\rho \sigma^2}{\eta^2} \left[ (\bar{y}_{n,t} - \mu) - \rho(\bar{y}_{n,t-1} - \mu) \right].
\end{align*}

The GA for risk is the same as in the static model for \( \rho = 0 \) [see Section i) and Example 4.1], since in this Gaussian framework the current factor \( f_t \) impacts the conditional distribution of \( m(F_{t+1}) = F_{t+1} \) given \( F_t = f_t \) through the mean only, and \( \sigma^2(F_{t+1}) = \sigma^2 \) is constant. The GA for filtering depends on both the risk level \( \alpha \) and the information through \( (\bar{y}_{n,t} - \mu) - \rho(\bar{y}_{n,t-1} - \mu) \).

By the latter effect, \( GA_{\text{filt}}(\alpha) \) can take any sign. Moreover, this term induces a stabilization effect on the dynamics of the GA VaR compared to the CSA VaR. To see this, let us assume \( \rho > 0 \) and suppose there is a large upward aggregate shock on the individual risks at date \( t \), such that \( \bar{y}_{n,t} - \mu \) is positive and (much) larger than \( \rho(\bar{y}_{n,t-1} - \mu) \). The CSA VaR in (5.12) reacts linearly to the shock and features a sharp increase. Since \( (\bar{y}_{n,t} - \mu) - \rho(\bar{y}_{n,t-1} - \mu) > 0 \), the GA term for filtering is negative and reduces the reaction of the VaR.

In Figure 2 we display the patterns of the true, CSA and GA VaR curves as a function of the risk level for a specific choice of parameters.

[Insert Figure 2: The VaR as a function of the risk level in the linear RFM with AR(1) factor.]

The mean and the autoregressive coefficient of the factor are \( \mu = 0 \) and \( \rho = 0.5 \), respectively. The idiosyncratic and systematic variance parameters \( \sigma^2 \) and \( \eta^2 \) are selected in order to imply an unconditional standard deviation of the individual risks \( \sqrt{\eta^2/(1 - \rho^2) + \sigma^2} = 0.15 \), and an unconditional correlation between individual risks \( \frac{\eta^2/(1 - \rho^2)}{\sigma^2 + \eta^2/(1 - \rho^2)} = 0.10 \). The portfolio size is \( n = 100 \). The available information \( I_{n,t} \) is such that \( \bar{y}_{n,t-j} = \mu = 0 \) for all lags \( j \geq 2 \), and we consider four different cases concerning the current and the most recent lagged cross-sectional averages, \( \bar{y}_{n,t} \) and \( \bar{y}_{n,t-1} \), respectively. Let us first assume \( \bar{y}_{n,t} = \bar{y}_{n,t-1} = 0 \) (upper-left Panel), that is, both cross-sectional averages are equal to the unconditional mean. As expected, all VaR curves are increasing w.r.t. the confidence level. The true VaR is about 0.10 at confidence level 99%. The CSA VaR underestimates the true
VaR (that is, underestimates the risk) by about 0.01 uniformly in the risk level. The GA for risk corrects most of this bias and dominates the GA for filtering. The situation is different when $\bar{y}_{n,t} = -0.30$ and $\bar{y}_{n,t-1} = 0$ (upper-right Panel), that is, when we have a downward aggregate shock in risk of two standard deviations at date $t$. The CSA VaR underestimates the true VaR by about 0.02. The GA for risk corrects only a rather small part of this bias, while including the GA for filtering allows for a quite accurate approximation. The GA for filtering is about five times larger than the GA for risk. When $\bar{y}_{n,t} = 0.30$ and $\bar{y}_{n,t-1} = 0$ (lower-left Panel), there is a large upward aggregate shock in risk at date $t$, and the CSA VaR overestimates the true VaR (that is, overestimates the risk). The GA correction for risk further increases the VaR and the bias, while including the GA correction yields a good approximation of the true VaR. The results are similar in the case $\bar{y}_{n,t} = 0.30$ and $\bar{y}_{n,t-1} = 0.30$ (lower-right Panel), that is, in case of a persistent downward aggregate shock in risk. Finally, by comparing the four panels in Figure 2, it is seen that the CSA risk measure is more sensitive to the current information than the true VaR and the GA VaR. To summarize, Figure 2 shows that the CSA VaR can either underestimate or overestimate the risk, the GA for filtering can dominate the GA for risk, and the complete GA can yield a good approximation of the true VaR even for portfolio sizes of some hundreds of contracts (at least in the specific linear RFM considered in our illustration). Moreover, the GA component reduces the undue time instability of the CSA VaR.

6 Stochastic Probability of Default and Expected Loss Given Default

6.1 Two-factor dynamic model

Let us consider a two-factor dynamic model with stochastic probability of default and expected loss given default. We extend Example 4.6 in Section 4.2 to a dynamic framework. The future value of the zero-coupon corporate bond maturing at $t + 1$ is:

$$y_{n,t+1} = LGD_{t+1} Z_{t+1},$$

where $Z_{t+1}$ is the default indicator and $LGD_{t+1}$ is the loss given default. Conditional on the path of the bivariate factor $F_t = (F_{1,t}, F_{2,t})'$, the default
indicator \( Z_{i,t+1} \) and the loss given default \( LGD_{i,t+1} \) are independent, such that \( Z_{i,t+1} \sim B(1, F_{1,t+1}) \) and \( LGD_{i,t+1} \) admits a beta distribution with conditional mean and volatility given by:

\[
E[LGD_{i,t+1}|F_{t+1}] = F_{2,t+1}, \quad V[LGD_{i,t+1}|F_{t+1}] = \gamma F_{2,t+1}(1 - F_{2,t+1}),
\]

where the concentration parameter \( \gamma \in (0,1) \) is constant. The dynamic factors \( F_{1,t} \) and \( F_{2,t} \) correspond to the (conditional) Probability of Default (PD) and the (conditional) Expected Loss Given Default (ELGD), respectively.

Both stochastic factors \( F_{1,t} \) and \( F_{2,t} \) admit values in the interval \((0,1)\). We assume that the transformed factors \( F^*_l = (F^*_{1,t}, F^*_{2,t})' \) defined by \( F^*_{l,t} = \log[F_{l,t} / (1 - F_{l,t})] \), for \( l = 1, 2 \) (logistic transformation), follow a bivariate Gaussian VAR(1) process:

\[
F^*_t = c + \Phi F^*_{t-1} + \varepsilon_t,
\]

with \( \varepsilon_t \sim \mathcal{N}(0, \Omega) \) and \( \Omega = \left( \begin{array}{cc} \sigma^2_1 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{array} \right) \).

6.2 CSA VaR and Granularity Adjustment

The CSA VaR and the GA are derived from the results in Section 5.2. Let us first consider the cross-sectional factor approximations. It is proved in Appendix 4 that:

\[
\hat{f}_{1,n,t} = \frac{N_t}{n},
\]

where \( N_t = \sum_{i=1}^{n} \mathbb{I}_{y_{i,t} > 0} \) is the number of defaults at date \( t \), and:

\[
\hat{f}_{2,n,t} = \arg \max_{f_{2,t}} \left\{ f_{2,t} \left( \frac{1 - \gamma}{\gamma} \right) \sum_{i:y_{i,t} > 0} \log \left( \frac{y_{i,t}}{1 - y_{i,t}} \right) \right. \\
- N_t \log \Gamma \left( \frac{1 - \gamma}{\gamma} f_{2,t} \right) - N_t \log \Gamma \left( \frac{1 - \gamma}{\gamma} (1 - f_{2,t}) \right) \left\}, \right.
\]

where the sum is over the companies that default at date \( t \), and \( \Gamma(.) \) denotes the Gamma function. Thus, the approximation \( \hat{f}_{1,n,t} \) of the conditional PD
is the cross-sectional default frequency at date \( t \), while the approximation \( \hat{f}_{2,n,t} \) of the conditional ELGD is obtained by maximizing the cross-sectional likelihood associated with the conditional beta distribution of the LGD at date \( t \). Proposition 3 on the approximate filtering distribution can be easily extended to multiple factor. Since the cross-sectional log-likelihood can be written as the sum of a component involving \( f_{1,t} \) only, and a component involving \( f_{2,t} \) only, the approximate filtering distribution is such that \( F_{1,t} \) and \( F_{2,t} \) are independent conditional on information \( I_{n,t} \) at order \( 1/n \), with Gaussian distributions \( \mathcal{N}\left( \hat{f}_{l,n,t} + \frac{1}{n} \mu_{l,n,t}, \frac{1}{n} J_{l,n,t}^{-1} \right) \), for \( l = 1, 2 \), where:

\[
\mu_{1,n,t} = -e_1' \Omega^{-1} (\hat{f}_{n,t}^* - c - \Phi \hat{f}_{n,t-1}^*),
\]

(5.15)

\[
\mu_{2,n,t} = -\frac{J_{2,n,t}^{-1}}{f_{2,n,t}(1 - \hat{f}_{2,n,t})} \left[ e_2' \Omega^{-1} (\hat{f}_{n,t}^* - c - \Phi \hat{f}_{n,t-1}^*) + 1 - 2 \hat{f}_{2,n,t} \right] + \frac{1}{2} J_{2,n,t}^{-1} K_{2,n,t},
\]

with \( \hat{f}_{n,t}^* = \left( \log \left[ \hat{f}_{1,n,t} / (1 - \hat{f}_{1,n,t}) \right], \log \left[ \hat{f}_{2,n,t} / (1 - \hat{f}_{2,n,t}) \right] \right)' \) and vectors \( e_1 = (1, 0)' \), \( e_2 = (0, 1)' \), and:

\[
J_{1,n,t} = \frac{1}{f_{1,n,t}(1 - \hat{f}_{1,n,t})},
\]

(5.16)

\[
J_{2,n,t} = \hat{f}_{1,n,t} \left( \frac{1 - \gamma}{\gamma} \right)^2 \left\{ \Psi' \left[ \left( \frac{1 - \gamma}{\gamma} \right) \hat{f}_{2,n,t} \right] + \Psi' \left[ \left( \frac{1 - \gamma}{\gamma} \right) (1 - \hat{f}_{2,n,t}) \right] \right\},
\]

with \( \Psi(s) = \frac{d \log \Psi(s)}{ds} \) and:

\[
K_{2,n,t} = -\hat{f}_{1,n,t} \left( \frac{1 - \gamma}{\gamma} \right)^3 \left\{ \Psi'' \left[ \left( \frac{1 - \gamma}{\gamma} \right) \hat{f}_{2,n,t} \right] - \Psi'' \left[ \left( \frac{1 - \gamma}{\gamma} \right) (1 - \hat{f}_{2,n,t}) \right] \right\}.
\]

(5.17)

Let us now derive the CSA VaR and the GA. From the results in Section 5.2 iv), we get the next Proposition.

**Proposition 6:** In the model with stochastic conditional PD and ELGD:

i) The CSA VaR at risk level \( \alpha \) is the solution of the equation:

\[
a \left( \text{VaR}_\infty(\alpha; \hat{f}_{n,t}), 0; \hat{f}_{n,t} \right) = \alpha,
\]

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where:

\[ a(w, 0; f_t) = \Phi \left( \frac{\log[w/(1-w)] - c_{1,t}}{\sigma_1} \right) \]

\[ + \int_w^1 \Phi \left[ \frac{\log[w/(y-w)] - c_{2,t} - \frac{\rho \sigma_2}{\sigma_1} (\log[y/(y-1)] - c_{1,t})}{\sigma_2 \sqrt{1-\rho^2}} \right] \cdot \frac{1}{\sigma_1} \varphi \left( \frac{\log[y/(y-1)] - c_{1,t}}{\sigma_1} \right) \frac{1}{y(1-y)} dy. \]

and \( c_{1,t} = c_1 + \Phi_{l,1} \log[f_{1,t}/(1 - f_{1,t})] + \Phi_{l,2} \log[f_{2,t}/(1 - f_{2,t})] \), for \( l = 1, 2 \).

ii) The GA at risk level \( \alpha \) is \( GA(\alpha) = GA_{\text{risk}}(\alpha) + GA_{\text{filt}}(\alpha) \), where \( GA_{\text{risk}}(\alpha) \) is computed from Proposition 4 with \( g_\infty(w; f_t) = b(w, 0; f_t) \) and:

\[ E[\sigma^2(F_{t+1}) | m(F_{t+1}) = w, F_t = f_t] = w(\gamma - w) + (1 - \gamma)w \frac{b(w, 1; f_t)}{b(w, 0; f_t)}, \]

where:

\[ b(w, k; f_t) = \int_w^1 \frac{y^{k-1}}{(1-y)w(1-w/y)} dy, \]

and \( \varphi(\cdot; \cdot; \rho) \) denotes the pdf of the standard bivariate Gaussian distribution with correlation \( \rho \); the component \( GA_{\text{filt}}(\alpha) \) is given by:

\[ GA_{\text{filt}}(\alpha) = - \frac{1}{g_\infty[VaR_\infty(\alpha; \hat{f}_{n,t}); \hat{f}_{n,t}]} \sum_{l=1}^2 \left\{ \frac{\partial a[VaR_\infty(\alpha; \hat{f}_{n,t}), 0; \hat{f}_{n,t}]}{\partial f_{l,t}} \mu_{l,n,t} \right. \]

\[ + \left. \frac{1}{2} J_{l,n,t}^{-1} \frac{\partial^2 a[VaR_\infty(\alpha; \hat{f}_{n,t}), 0; \hat{f}_{n,t}]}{\partial f_{l,t}^2} \right\}, \]

where \( \mu_{l,n,t} \) and \( J_{l,n,t} \) are given in (5.15) and (5.16).

**Proof:** See Appendix 4.

The CSA and GA VaR in Proposition 6 are given in closed form, up to a few one-dimensional integrals. [TO BE CONTINUED]
7 Concluding Remarks

For large homogenous portfolios and a variety of both single-factor and multifactor dynamic risk models, closed form expressions of the VaR and other distortion risk measures can be derived at order $1/n$. Two granularity adjustments are required. The first GA concerns the conditional VaR with current factor value assumed to be observed. The second GA takes into account the unobservability of the current factor value and is specific to dynamic factor models. This explains why this GA has not been taken into account in the earlier literature which focuses on static models.

These GA assume given the link function $c$ and the distributions of both the factor and idiosyncratic risks. In practice the link function and the distribution depend on unknown parameters, which have to be estimated. This creates an additional error on the VaR, which has been considered neither here, nor in the previous literature. This estimation error can be larger than the GA derived in this paper. However, such a separate analysis is compatible with the Basel 2 methodology. Indeed, the GA in this paper are useful to compute the reserves for Credit Risk, whereas the adjustment for estimation concerns the reserves for Estimation Risk.

The granularity adjustment principle appeared in Pillar 1 of the New Basel Capital Accord in 2001 [BCBS (2001)], concerning the minimum capital requirement. It has been suppressed from Pillar 1 in the most recent version of the Accord in 2003 [BCBS (2003)], and assigned to Pillar 2 on internal risk models. The recent financial crisis has shown that systematic risks, which include in particular systemic risks\(^\text{16}\), have to be distinguished from unsystematic risk, and in the new organization these two risks will be supervised by two different regulators. This shows the importance of taking into account this distinction in computing the reserves, that is, also at Pillar 1 level. For instance, one may fix different risk levels $\alpha_1$ and $\alpha_2$ in the CSA and GA VaR components, and smooth differently these components over the cycle in the definition of the required capital. The recent literature on granularity shows that the technology is now in place and rather easy to implement, at least for nonlinear static, and linear dynamic, factor models.

\(^{16}\)A systemic risk is a systematic risk which can seriously damage the Financial System.


Figure 1: CreditVaR as a function of the confidence level in the Merton-Vasicek model.

In each Panel we display the CSA VaR (dashed line) and the GA VaR (solid line) of the default frequency in a portfolio of \( n = 100 \) companies, as a function of the confidence level \( \alpha \). The four panels correspond to different values of the unconditional probability of default \( PD \) and asset correlation \( \rho \), that are \( PD = 0.005 \) and \( \rho = 0.12 \) in the upper-left panel, \( PD = 0.005 \) and \( \rho = 0.24 \) in the upper-right panel, \( PD = 0.05 \) and \( \rho = 0.12 \) in the lower-left panel, and \( PD = 0.05 \) and \( \rho = 0.24 \) in the lower-right panel, respectively.
Figure 2: VaR as a function of the risk level in the linear RFM with AR(1) factor.

In each Panel we display the true VaR (solid line), the CSA VaR (dashed line), the GA VaR accounting for risk only (dashed-dotted line) and the GA VaR accounting for both risk and filtering (dotted line), as a function of the confidence level $\alpha$. The four Panels correspond to different available informations $I_{n,t}$, that are $\bar{y}_{n,t} = \bar{y}_{n,t-1} = 0$ in the first Panel, $\bar{y}_{n,t} = -0.30$, $\bar{y}_{n,t-1} = 0$ in the second, $\bar{y}_{n,t} = 0.30$, $\bar{y}_{n,t-1} = 0$ in the third, and $\bar{y}_{n,t} = 0.30$, $\bar{y}_{n,t-1} = 0.30$ in the fourth Panel, respectively. The portfolio size is $n = 100$. The model parameters are such that the unconditional standard deviation of the individual risks is 0.15, the unconditional correlation between individual risks is 0.10, the factor mean is $\mu = 0$, and the factor autoregressive coefficient is $\rho = 0.5$. 

Appendix 1

Asymptotic Expansions

1. Expansion of the cumulative distribution function

Let us consider a pair \((X, Y)\) of real random variables, where \(X\) is a continuous random variable with pdf \(f_1\) and cdf \(G_1\). The aim of this section is to derive an expansion of the function:

\[ a(x, \varepsilon) = P[X + \varepsilon Y < x], \quad (a.1) \]

in a neighbourhood of \(\varepsilon = 0\).

The following Lemma has been first derived by Gouriéroux, Laurent, Scaillet (2000) for the second-order expansion, and by Martin, Wilde (2002) for any order. We will extend the proof in Gouriéroux, Laurent, Scaillet (2000), which avoids the use of characteristic functions and shows more clearly the needed regularity conditions (RC) (see below).

Lemma a.1: Under regularity conditions (RC), we have:

\[ a(x, \varepsilon) = G_1(x) + \sum_{j=1}^{J} \left\{ \frac{(-1)^j}{j!} \varepsilon^j d^{j-1} \left[ g_1(x) E(Y^j | X = x) \right] \right\} + o(\varepsilon^J). \]

Proof: The proof requires two steps. First, we consider the case of a bivariate continuous random vector \((X, Y)\). Then, we extend the result when \(Y\) and \(X\) are in a deterministic relationship.

i) Bivariate continuous vector

Let us denote by \(g_{1,2}(x, y)\) [resp. \(g_{1/2}(x|y)\) and \(G_{1/2}(x|y)\)] the joint pdf of \((X, Y)\) (resp. the conditional pdf and cdf of \(X\) given \(Y\)). We have:
\[ a(x, \varepsilon) = P[X + \varepsilon Y < x] \]
\[ = EP[X < x - \varepsilon Y | Y] \]
\[ = E[G_{1|2}(x - \varepsilon Y | Y)] \]
\[ = E[G_{1|2}(x | Y)] + \sum_{j=1}^{J} \left\{ \frac{(-1)^{j} \varepsilon^j}{j!} E[Y^j \frac{d^{j-1}}{dx^{j-1}} g_{1|2}(x | Y)] \right\} + o(\varepsilon^J) \]
\[ = G_1(x) + \sum_{j=1}^{J} \left\{ \frac{(-1)^{j} \varepsilon^j}{j!} \frac{d^{j-1}}{dx^{j-1}} (E[Y^j g_{1|2}(x | Y)]) \right\} + o(\varepsilon^J). \]

ii) Variables in deterministic relationship

Let us now consider the case of a function \( a(x, \varepsilon) = P[X + \varepsilon c(X) < x] \), where the direction of expansion \( Y = c(X) \) is in a deterministic relationship with variable \( X \). Let us introduce a variable \( Z \) independent of \( X \) with a gamma distribution \( \gamma(\nu, \nu) \), and study the function:
\[ a(x, \varepsilon; \nu) = P[X + \varepsilon c(X)Z < x]. \]

The joint distribution of \( [X, Y^* = c(X)Z] \) is continuous; thus, the results of part i) of the proof can be applied. We get:
\[ a(x, \varepsilon; \nu) = G_1(x) + \sum_{j=1}^{J} \left\{ \frac{(-1)^{j} \varepsilon^j}{j!} \frac{d^{j-1}}{dx^{j-1}} (g_1(x)E[c(X)^j Z^j | X = x]) \right\} + o(\varepsilon^J) \]
\[ = G_1(x) + \sum_{j=1}^{J} \left\{ \frac{(-1)^{j} \varepsilon^j}{j!} \frac{d^{j-1}}{dx^{j-1}} [g_1(x)E[c(X)^j Z^j]] \right\} + o(\varepsilon^J) \]
\[ = G_1(x) + \sum_{j=1}^{J} \left\{ \frac{(-1)^{j} \varepsilon^j}{j!} \frac{d^{j-1}}{dx^{j-1}} [g_1(x)c(x)^j E[Z^j]] \right\} + o(\varepsilon^J). \]
where:

\[ \mu_j(\nu) = E(Z^j) = (1 - \frac{1}{\nu})(1 - \frac{2}{\nu}) \ldots (1 - \frac{j-1}{\nu}), \quad j = 1, \ldots, J. \]

Since the moments \( \mu_j(\nu), j = 1, \ldots, J \) tend uniformly to 1, and \( Y^* = Zc(X) \) tends to \( Y = c(X) \), when \( \nu \) tends to infinity, we get:

\[
a(x, \varepsilon) = P[X + \varepsilon c(X) < x] = \lim_{\nu \to \infty} a(x, \varepsilon; \nu) = G_1(x) + \sum_{j=1}^{J} \left\{ \frac{(-1)^j \varepsilon^j}{j!} \frac{d^{j-1}}{dx^{j-1}} [g_1(x)c(x)^j] \right\} + o(\varepsilon^J).
\]

QED

2. Application to large portfolio risk

Let us consider the asymptotic expansion (3.2):

\[
W_n/n = m(F) + \sigma(F)X/\sqrt{n} + O(1/n),
\]
where \( X \) is independent of \( F \) with distribution \( N(0,1) \) and the term \( O(1/n) \) is zero-mean, conditional on \( F \). We have:

\[
a_n(x) = P[W_n/n < x] = P[m(F) + \sigma(F)X/\sqrt{n} + O(1/n) < x].
\]

Then, the expansion in Lemma a.1 can be applied at order 2, noting that:

\[
E[O(1/n)|F] = 0, \quad E[\sigma(F)X|m(F)] = E[\sigma(F)|m(F)]E[X] = 0,
\]

\[
E[\sigma^2(F)X^2|m(F)] = E[X^2]E[\sigma^2(F)|m(F)] = E[\sigma^2(F)|m(F)].
\]

We deduce the following Lemma:

**Lemma a.2:** We have:

\[
a_n(x) = P[W_n/n < x] = P[m(F) < x] + \frac{1}{2n} \frac{d}{dx} \left\{ g_\infty(x)E[\sigma^2(F)|m(F) = x] \right\} + o(1/n),
\]

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where $g_\infty$ is the pdf of $m(F)$.

Note that this approximation at order $1/n$ is exact and not itself approximated by the cdf based on a bivariate distribution as in Vasicek (2002).

3. Expansion of the VaR

The expansion of the VaR per individual is deduced from the Bahadur’s expansion [see Bahadur (1966), or Gagliardini, Gouriéroux (2010), Section 6.2].

Lemma a.3 (Bahadur’s expansion): Let us consider a sequence of cdf’s $F_n$ tending to a limiting cdf $F_\infty$ at uniform rate $1/n$:

$$F_n(x) = F_\infty(x) + O(1/n).$$

Let us denote by $Q_n$ and $Q_\infty$ the associated quantile functions and assume that the limiting distribution is continuous with density $f_\infty$. Then:

$$Q_n(\alpha) - Q_\infty(\alpha) = -\frac{F_n[Q_\infty(\alpha)] - F_\infty[Q_\infty(\alpha)]}{f_\infty[Q_\infty(\alpha)]} + o(1/n).$$

Lemma a.3 can be applied to the standardized portfolio risk. The limiting distribution is the distribution of $m(F)$ with pdf $g_\infty$ and quantile function $VaR_\infty$. By using the expansion of Lemma a.2, we get:

$$VaR_n(\alpha) = VaR_\infty(\alpha) - \frac{1}{2n} \frac{1}{g_\infty[VaR_\infty(\alpha)]} \left[ \frac{d}{dx} \left\{ g_\infty(x) E[\sigma^2(F)|m(F) = x] \right\} \right]_{x=VaR_\infty(\alpha)} + o(1/n)$$

$$= VaR_\infty(\alpha) - \frac{1}{2n} \left\{ \frac{d \log g_\infty}{dx}[VaR_\infty(\alpha)] E[\sigma^2(F)|m(F) = VaR_\infty(\alpha)] \right\} + o(1/n).$$

This is the result in Proposition 1.
Appendix 2: GA when $m(F)$ is monotone increasing

From Proposition 1 we have:

$$GA(\alpha) = -\frac{1}{2} \left( \frac{1}{g_\infty(z)} \frac{d}{dz} \left\{ g_\infty(z) E\left[ \sigma^2(F) | m(F) = z \right] \right\} \right)_{z=\text{VaR}_\infty(\alpha)}$$

$$= -\frac{1}{2} \left( \frac{1}{g_\infty(z)} \frac{d}{dz} \left\{ g_\infty(z) \sigma^2[m^{-1}(z)] \right\} \right)_{z=\text{VaR}_\infty(\alpha)}$$

$$= -\frac{1}{2} \left( \frac{dm}{dF}[m^{-1}(z)] \frac{d}{dm} \left\{ \tilde{g}_\infty[m^{-1}(z)] \sigma^2[m^{-1}(z)] \right\} \right)_{z=\text{VaR}_\infty(\alpha)}$$

$$= -\frac{1}{2} \left( \frac{1}{\tilde{g}_\infty[m^{-1}(z)]} \frac{d}{dm} \left\{ \tilde{g}_\infty(m^{-1}(z)) \sigma^2[m^{-1}(z)] \right\} \right)_{z=\text{VaR}_\infty(\alpha)}$$

$$= -\frac{1}{2} \left( \frac{1}{\tilde{g}_\infty(f)} \frac{d}{df} \left\{ \tilde{g}_\infty(f) \sigma^2(f) \right\} \right)_{f=m^{-1}[\text{VaR}_\infty(\alpha)]}.$$
Appendix 4
Stochastic probability of default and expected loss given default

In this Appendix we give a detailed derivation of the granularity adjustment in the model with stochastic probability of default and expected loss given default presented in Section 6.

i) Cross-sectional factor approximation

Let us consider date $t$ and assume that we observe the individual losses $y_{i,t}$, $i = 1, \ldots, n$, of the zero-coupon corporate bonds maturing at date $t$. Equivalently, we observe the default indicator $Z_{i,t} = 1_{y_{i,t} > 0}$ for all individual companies, and the Loss Given Default $LGD_{i,t} = y_{i,t}$ if $Z_{i,t} = 1$. Thus, the model is equivalent to a truncated model and the cross-sectional likelihood conditional on the factor value is:

$$
\prod_{i=1}^{n} h(y_{i,t} | f_t) = \prod_{i=1}^{n} f_{1,t}^{Z_{i,t}} (1 - f_{1,t})^{1 - Z_{i,t}} \prod_{i: Z_{i,t} = 1} \frac{\Gamma(a_t + b_t)}{\Gamma(a_t) \Gamma(b_t)} LGD_{i,t}^{a_t - 1} (1 - LGD_{i,t})^{b_t - 1}
$$

where $N_t = \sum_{i=1}^{n} 1_{y_{i,t} > 0}$ is the number of defaults at date $t$, and $a_t = \left(\frac{1 - \gamma}{\gamma}\right) f_{2,t}$ and $b_t = \left(\frac{1 - \gamma}{\gamma}\right) (1 - f_{2,t})$. We get the cross-sectional log-likelihood:

$$
\sum_{i=1}^{n} \log h(y_{i,t} | f_t) = N_t \log f_{1,t} + (n - N_t) \log (1 - f_{1,t}) + N_t \log \left[\frac{1 - \gamma}{\gamma}\right] - N_t \log \left[\left(\frac{1 - \gamma}{\gamma}\right) f_{2,t}\right] - N_t \log \left[\left(\frac{1 - \gamma}{\gamma}\right) (1 - f_{2,t})\right] + \left[\left(\frac{1 - \gamma}{\gamma}\right) f_{2,t} - 1\right] \sum_{i: y_{i,t} > 0} \log y_{i,t} + \left[\left(\frac{1 - \gamma}{\gamma}\right) (1 - f_{2,t}) - 1\right] \sum_{i: y_{i,t} > 0} \log (1 - y_{i,t})
$$

$$
= \mathcal{L}_{1,n,t}(f_{1,t}) + \mathcal{L}_{2,n,t}(f_{2,t}), \quad \text{say.} \quad \text{(a.2)}
$$
which is decomposed as the sum of a function of \( f_{1,t} \) and a function of \( f_{2,t} \). Therefore, the cross-sectional approximations of the two factors can be computed separately, and we get (5.13) and (5.14).

ii) Approximate filtering distribution of \( f_t \) given \( I_{n,t} \)

From a multiple factor version of Corollary 5.3 in Gagliardini, Gouriéroux (2009), and the log-likelihood decomposition in (a.2), it follows that the approximate filtering distribution is such that \( F_{1,t} \) and \( F_{2,t} \) are independent conditional on information \( I_{n,t} \) at order 1/n, with Gaussian distributions \( N \left( \hat{f}_{l,n,t} + \frac{1}{n} \mu_{l,n,t}, \frac{1}{n} J_{l,n,t}^{-1} \right) \), for \( l = 1, 2 \), where:

\[
\mu_{l,n,t} = J_{l,n,t}^{-1} \frac{\partial \log g}{\partial f_{l,t}} (\hat{f}_{n,t} | \hat{f}_{n,t-1}) + \frac{1}{2} J_{l,n,t}^{-2} K_{l,n,t},
\]

\[
J_{l,n,t} = - \frac{1}{n \sum_{i=1}^{n} \frac{\partial^2 \log h}{\partial f_{l,t}^2} (y_{i,t} | \hat{f}_{n,t})} = - \frac{1}{n} \frac{\partial^2 \mathcal{L}_{l,n,t}}{\partial f_{l,t}^2} (\hat{f}_{l,n,t}),
\]

\[
K_{l,n,t} = \frac{1}{n \sum_{i=1}^{n} \frac{\partial^3 \log h}{\partial f_{l,t}^3} (y_{i,t} | \hat{f}_{n,t})} = \frac{1}{n} \frac{\partial^3 \mathcal{L}_{l,n,t}}{\partial f_{l,t}^3} (\hat{f}_{l,n,t}).
\]

From (a.2), equations (5.16) and (5.17) follow, as well as \( K_{1,n,t} = 2 \frac{1 - 2 \hat{f}_{1,n,t}}{\hat{f}_{1,n,t}^2 (1 - \hat{f}_{1,n,t})^2} \). Moreover:

\[
\frac{\partial \log g}{\partial f_{l,t}} (f_t | f_{t-1}) = - \frac{1}{a[A^{-1}(f_{l,t})]} \left\{ \epsilon'_t \left[ \Omega^{-1}(f_t^* - c - \Phi f_{t-1}^*) \right] + \frac{d \log a}{dy} [A^{-1}(f_{l,t})] \right\}
\]

\[
= - \frac{1}{f_{l,t} (1 - f_{l,t})} \left\{ \epsilon'_t \left[ \Omega^{-1}(f_t^* - c - \Phi f_{t-1}^*) \right] + 1 - 2 f_{l,t} \right\},
\]

for \( l = 1, 2 \), where \( A(y) = [1 + \exp(-y)]^{-1} \) and \( a(y) = dA(y)/dy \). Then equations (5.15) follow.

iii) A useful lemma

The derivation of the CSA VaR and the GA below uses the next lemma.
Lemma a.4: Let $X$ and $Y$ be random variables on $[0, 1]$. Denote by $f(x,y)$, $f_2(y)$, $F_2(y)$ and $F_{1|2}(x|y)$ the joint pdf of $(X,Y)$, the pdf of $Y$, the cdf of $Y$ and the conditional cdf of $X$ given $Y$, respectively. Let $Z = XY$. Then:

(i) $P[Z \leq z] = \int_z^1 F_{1|2}(z/y|y)f_2(y)dy + F_2(z)$, for any $z \in [0, 1]$;

(ii) The pdf of $Z$ is $g(z) = \int_z^1 f(z/y, y)dy$, $z \in [0, 1]$;

(iii) $E[Y|Z = z] = \frac{\int_z^1 f(z/y, y)dy}{\int_z^1 f(z/y, y)dy}$, for any $z \in [0, 1]$.

Proof: (i) We have:

$$P[Z \leq z] = EP[Z \leq z|Y] = EP[X \leq z/Y|Y].$$

Now:

$$P[X \leq z/Y|Y] = \mathbb{I}_{z \leq Y} F_{1|2}(z/Y|Y) + \mathbb{I}_{z > Y}.$$

Thus, we get:

$$P[Z \leq z] = \int_z^1 F_{1|2}(z/y|y)f_2(y)dy + F_2(z).$$

(ii) By differentiating the cdf found in (i) we get:

$$g(z) = \frac{d}{dz} \left( \int_z^1 F_{1|2}(z/y|y)f_2(y)dy + F_2(z) \right)$$

$$= \int_z^1 \frac{1}{y} f(z/y, y)f_2(y)dy - \left[F_{1|2}(z/y|y)f_2(y)\right]_{y=z} + f_2(z)$$

$$= \int_z^1 \frac{1}{y} f(z/y, y)dy.$$  

(iii) Let us consider the change of variables from $(x, y)$ to $(z, y)$, where $z = xy$. The Jacobian is $\left| \text{det} \begin{bmatrix} \frac{\partial(z,y)}{\partial(x,y)} \end{bmatrix} \right| = y$. Thus, the joint density of $Z$ and $Y$ is
\( g(z, y) = \frac{1}{y} f(z/y, y), \) for \( 0 \leq z \leq y \leq 1. \) We get:

\[
E[Y|Z = z] = \frac{\int_z^1 yg(z, y)dy}{\int_z^1 g(z, y)dy} = \frac{\int_z^1 f(z/y, y)dy}{\int_z^1 y f(z/y, y)dy}.
\]

QED

iv) CSA risk measure

Let us first compute function \( a(w, 0; f) \). From the dynamic version of equation (4.13), we have:

\[
m(F_{t+1}) = F_{1,t+1} F_{2,t+1}.
\]

Then:

\[
a(w, 0; f_t) = P[m(F_{t+1}) \leq w|F_t = f_t] = P[F_{1,t+1} F_{2,t+1} \leq w|F_t = f_t].
\]

We apply Lemma a.4 (i) with \( X = F_{2,t+1} \) and \( Y = F_{1,t+1} \), conditionally on \( F_t = f_t \). The distribution of \( F_{2,t+1}^* \) conditional on \( F_{1,t+1}^* \) and \( F_t = f_t \) is:

\[
N \left( c_{2,t} + \frac{\rho \sigma_2}{\sigma_1} (F_{1,t+1}^* - c_{1,t}), \sigma_2^2 (1 - \rho^2) \right),
\]

where \( c_{1,t} = c_1 + \Phi_{11} A^{-1}(f_{1,t}) + \Phi_{12} A^{-1}(f_{2,t}) \) and \( c_{2,t} = c_2 + \Phi_{21} A^{-1}(f_{1,t}) + \Phi_{22} A^{-1}(f_{2,t}) \). Thus, the conditional cdf of \( F_{2,t+1} \) given \( F_{1,t+1} = y \) and \( F_t = f_t \) is:

\[
P[F_{2,t+1} \leq x|F_{1,t+1} = y, F_t = f_t] = \Phi \left[ \frac{A^{-1}(x) - c_{2,t} - \frac{\rho \sigma_2}{\sigma_1} (A^{-1}(y) - c_{1,t})}{\sigma_2 \sqrt{1 - \rho^2}} \right],
\]

for \( x, y \in (0, 1) \). The cdf and the pdf of \( F_{1,t+1} \) conditional on \( F_t = f_t \) are:

\[
F(y|f_t) = \Phi \left( \frac{A^{-1}(y) - c_{1,t}}{\sigma_1} \right), \quad f(y|f_t) = \frac{1}{\sigma_1} \varphi \left( \frac{A^{-1}(y) - c_{1,t}}{\sigma_1} \right) \frac{1}{a[A^{-1}(y)]},
\]

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respectively, where \(a(y) = dA(y)/dy\). Thus, from Lemma a.4 we get:

\[
a(w, 0; ft) = \Phi \left( \frac{A^{-1}(w) - c_1, t}{\sigma_1} \right)
- \int_w^1 \Phi \left[ \frac{A^{-1}(w/y) - c_2, t - \frac{\rho \sigma_2 (A^{-1}(y) - c_1, t)}{\sigma_2 \sqrt{1 - \rho^2}}}{\sigma_2 \sqrt{1 - \rho^2}} \right]
\cdot \frac{1}{\sigma_1} \varphi \left( \frac{A^{-1}(y) - c_1, t}{\sigma_1} \right) \frac{1}{a[A^{-1}(y)]} \, dy.
\]

By using that \(a[A^{-1}(y)] = y(1 - y)\) and \(A^{-1}(w/y) = \log[w/(y - w)]\), Proposition 6 i) follows.

v) Granularity adjustment

Let us first compute the density \(g_\infty(.; ft)\). We use equation (a.3) and apply Lemma a.4 (ii) with \(X = F_{1,t+1}\) and \(Y = F_{2,t+1}\) conditionally on \(F_t = f_t\). The joint density of \((X, Y)\) conditionally on \(F_t = f_t\) is:

\[
f(x, y) = \frac{1}{\sigma_1 \sigma_2} \varphi \left( \frac{A^{-1}(x) - c_1, t}{\sigma_1}, \frac{A^{-1}(y) - c_2, t}{\sigma_2}; \rho \right) \frac{1}{a[A^{-1}(x)]a[A^{-1}(y)]},
\]

where \(\varphi(., .; \rho)\) denotes the pdf of a bivariate standard Gaussian distribution with correlation parameter \(\rho\). We get:

\[
g_\infty(w; f_t) = \int_w^1 \frac{1}{y \sigma_1 \sigma_2} \varphi \left( \frac{A^{-1}(w/y) - c_1, t}{\sigma_1}, \frac{A^{-1}(y) - c_2, t}{\sigma_2}; \rho \right)
\cdot \frac{1}{a[A^{-1}(w/y)]a[A^{-1}(y)]} \, dy.
\]

Let us now compute \(E[\sigma^2(F_{t+1})|m(F_{t+1}) = w, F_t = f_t]\). From the dynamic version of equation (4.14), we have:

\[
\sigma(F_{t+1}) = \gamma F_{2,t+1}(1 - F_{2,t+1})F_{1,t+1} + F_{1,t+1}(1 - F_{1,t+1})F_{2,t+1},
\]

By replacing \(F_{t+1} F_{2,t+1}\) with \(m(F_{t+1})\) in equation (4.14), function \(\sigma^2(F_{t+1})\) can be rewritten as:

\[
\sigma^2(F_{t+1}) = \gamma (F_{1,t+1} F_{2,t+1})(1 - F_{2,t+1}) + (F_{1,t+1} F_{2,t+1})(F_{2,t+1} - F_{1,t+1} F_{2,t+1})
= m(F_{t+1})[\gamma - m(F_{t+1})] + (1 - \gamma)m(F_{t+1}) F_{2,t+1}.
\]
Thus:

\[
E [\sigma^2(F_{t+1})|m(F_{t+1}) = w, F_t = f_t] = w(\gamma - w)
+ (1 - \gamma)wE[F_{2,t+1}|F_{1,t+1}F_{2,t+1} = w, F_t = f_t].
\]

(a.4)

From Lemma a.4 (iii) with \(X = F_{1,t+1}\) and \(Y = F_{2,t+1}\), conditionally on \(F_t = f_t\) we get:

\[
E[F_{2,t+1}|m(F_{t+1}) = w, F_t = f_t] = \frac{b(w, 1; f_t)}{g_\infty(w; f_t)},
\]

where:

\[
b(w, 1; f_t) = \int_1^w \frac{1}{\sigma_1 \sigma_2} \phi \left( \frac{A^{-1}(w/y) - \mu_{1,t}}{\sigma_1}, \frac{A^{-1}(y) - \mu_{2,t}}{\sigma_2}, \rho \right) \frac{1}{a[A^{-1}(w/y)]a[A^{-1}(y)]} dy.
\]

Then, from equation (a.4) we get:

\[
E[\sigma^2(F_{t+1})|m(F_{t+1}) = w, F_t = f_t] = w(\gamma - w) + (1 - \gamma)w \frac{b(w, 1; f_t)}{g_\infty(w; f_t)}.
\]

By using \(a[A^{-1}(w/y)]a[A^{-1}(y)] = w(1 - y)(1 - w)(1 - w/y)\), Proposition 6 ii) follows.